Problems #8

Problem 1. (2 XP) Let $p \in \mathbb{Z}_{\geqslant 0}$. We have already seen that the sums of powers

$$S_n^{(p)} = 1^p + 2^p + \ldots + n^p$$

can be expressed in terms of Bernoulli polynomials. Let us consider an alternative approach here.

(a) Show that

$$\sum_{n=0}^{\infty} S_n^{(p)} x^n = \frac{1}{1-x} (x D)^p \frac{1}{1-x}.$$

(b) Use this identity to find (again) explicit formulas for $S_n^{(p)}$ in the cases $p = 1, 2, 3$.

(c) **(bonus challenge, 2 XP extra)** Can you generalize these to provide a general formula that holds for all p?

Problem 2. (3 XP) The *Dirichlet series generating function* of a sequence $(a_n)_{n \geqslant 1}$ is the function $\sum_{n \geqslant 1} \frac{a_n}{n^s}$.

(a) What is the Dirichlet series generating function of the sequence $(n^3)_{n \geqslant 1}$?

(b) Which sequence is generated by the Dirichlet series generating function $\zeta(s)^2$?

(c) For given λ, which sequence is generated by $\zeta(s)\zeta(s - \lambda)$?

(d) Suppose that a_n is fully multiplicative, that is, $a_{nm} = a_n a_m$ for all $n, m \in \mathbb{Z}_{\geqslant 1}$. Show that

$$\sum_{n \geqslant 1} \frac{a_n}{n^s} = \prod_p \left(1 - \frac{a_p}{p^s}\right)^{-1},$$

where the infinite product is over all primes p.

Problem 3. (2 XP) Let $N > 1$, and let $h = (b - a)/N$. In numerical analysis, the (composite) trapezoidal rule

$$\int_a^b f(x)\, \mathrm{d}x \approx h\left[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \ldots + f(b-h) + \frac{f(b)}{2}\right]$$

is used to approximate definite integrals.

(a) Show that the error of this approximation is $O(h^2)$ if $f \in C^2[a, b]$.

(b) Spell out the first, say, two terms of the asymptotic for the error under the assumption that f is sufficiently differentiable.

(c) **(1 XP extra)** The trapezoidal rule works amazingly well when the integrand $f(x)$ is smooth and periodic with period $b - a$. Can you explain why?

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