Problem 1. (2 XP) Let $p \in \mathbb{Z}_{\geqslant 0}$. We have already seen that the sums of powers

$$S_n^{(p)} = 1^p + 2^p + \ldots + n^p$$

can be expressed in terms of Bernoulli polynomials. Let us consider an alternative approach here.

(a) Show that

$$\sum_{n=0}^{\infty} S_n^{(p)} x^n = \frac{1}{1-x} (x D)^p \frac{1}{1-x}.$$

(b) Use this identity to find (again) explicit formulas for $S_n^{(p)}$ in the cases $p = 1, 2, 3$.

(c) **(bonus challenge, 2 XP extra)** Can you generalize these to provide a general formula that holds for all p?

Solution.

(a) Observe that $(x D)^p \frac{1}{1-x}$ is the generating function of $(n^p)_{n \geqslant 0}$. The claim now follows from the fact that, if $F(x)$ is the ogf of a_n, then $F(x)/(1-x)$ is the ogf of the partial sums $a_0 + a_1 + \ldots + a_n$.

(b) The basic ingredient for the computations will be the identity

$$\frac{1}{(1-x)^{k+1}} = \sum_{n \geqslant 0} \binom{n+k}{k} x^n,$$

which we derived earlier.

- In the case $p = 1$, we have

$$\frac{1}{1-x} (x D) \frac{1}{1-x} = \frac{x}{(1-x)^3},$$

and hence, extracting the coefficient of x^n,

$$\sum_{k=1}^{n} k = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

- In the case $p = 2$, we have

$$\frac{1}{1-x} (x D)^2 \frac{1}{1-x} = \frac{1}{1-x} (x^2 D^2 + x D) \frac{1}{1-x} = \frac{2x^2}{(1-x)^4} + \frac{x}{(1-x)^3},$$

and hence, extracting the coefficient of x^n,

$$\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{n(n+1)(2n+1)}{6}.$$

- In the case $p = 3$, we have

$$\frac{1}{1-x} (x D)^3 \frac{1}{1-x} = \frac{1}{1-x} (x^3 D^3 + 3x^2 D^2 + x D) \frac{1}{1-x} = \frac{6x^3}{(1-x)^5} + \frac{6x^2}{(1-x)^4} + \frac{x}{(1-x)^3},$$

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and hence, extracting the coefficient of x^n,

$$\sum_{k=1}^{n} k^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = \left[\frac{n(n+1)}{2}\right]^2.$$

Challenge. Can you identify the coefficients in front of the binomial coefficients?

By the way, by putting the rational function on a common denominator before extracting coefficients, we obtain different combinations of binomial coefficients. For instance,

$$\frac{1}{1-x}(x\,D)^3\frac{1}{1-x} = \frac{x^3+4x^2+x}{(1-x)^5} = \frac{x^3}{(1-x)^5} + \frac{4x^2}{(1-x)^5} + \frac{x}{(1-x)^5},$$

so that we get

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4} = \left[\frac{n(n+1)}{2}\right]^2.$$

Challenge. Again, can you identify these coefficients and produce a general formula? □

Problem 2. (3 XP) The *Dirichlet series generating function* of a sequence $(a_n)_{n\geqslant 1}$ is the function $\sum_{n\geqslant 1}\frac{a_n}{n^s}$.

(a) What is the Dirichlet series generating function of the sequence $(n^3)_{n\geqslant 1}$?

(b) Which sequence is generated by the Dirichlet series generating function $\zeta(s)^2$?

(c) For given λ, which sequence is generated by $\zeta(s)\zeta(s-\lambda)$?

(d) Suppose that $a(n)$ is fully multiplicative, that is, $a(nm)=a(n)a(m)$ for all $n,m\in\mathbb{Z}_{\geqslant 1}$. Show that

$$\sum_{n\geqslant 1}\frac{a(n)}{n^s} = \prod_p \left(1-\frac{a(p)}{p^s}\right)^{-1},$$

where the infinite product is over all primes p.

Solution.

(a) The generating function is $\sum_{n\geqslant 1}\frac{n^3}{n^s} = \zeta(s-3)$.

(b) We observe that Dirichlet series generating functions multiply according to

$$\left(\sum_{n\geqslant 1}\frac{a_n}{n^s}\right)\left(\sum_{n\geqslant 1}\frac{b_n}{n^s}\right) = \sum_{n\geqslant 1}\frac{c_n}{n^s}, \quad c_n = \sum_{d|n} a_d b_{n/d}.$$

In our case, we have $a_n=b_n=1$, so that the sequence generated by $\zeta(s)^2$ is the number of divisors of n.

(c) Note that, as in the first part of this problem,

$$\zeta(s-\lambda) = \sum_{n\geqslant 1}\frac{1}{n^{s-\lambda}} = \sum_{n\geqslant 1}\frac{n^\lambda}{n^s}$$

generates the sequence n^λ. Hence, by the second part, the sequence c_n generated by $\zeta(s)\zeta(s-\lambda)$ is the sum of powers of divisors

$$c_n = \sum_{d|n} d^\lambda,$$

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which is usually denoted by $\sigma_\lambda(n)$.

(d) Let us write Because n can be factored uniquely into prime powers $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, and because, in that case,

$$a(n) = a(p_1^{r_1}) a(p_2^{r_s}) \cdots a(p_s^{r_s}),$$

we have

$$\sum_{n \geqslant 1} \frac{a(n)}{n^s} = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \ldots\right).$$

So far, we have only used that $a(n)$ is multiplicative in the sense that $a(nm) = a(n)a(m)$ if n and m are coprime. If $a(n)$ is fully multiplicative, we further have

$$\sum_{n \geqslant 1} \frac{a(n)}{n^s} = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p)^2}{p^{2s}} + \ldots\right) = \prod_p \left(1 - \frac{a(p)}{p^s}\right)^{-1}. \qquad \square$$

Problem 3. (2 XP) Let $N > 1$, and let $h = (b - a)/N$. In numerical analysis, the (composite) trapezoidal rule

$$\int_a^b f(x)\,\mathrm{d}x \approx h\left[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \ldots + f(b-h) + \frac{f(b)}{2}\right]$$

is used to approximate definite integrals.

(a) Show that the error of this approximation is $O(h^2)$ if $f \in C^2[a, b]$.

(b) Spell out the first, say, two terms of the asymptotic for the error under the assumption that f is sufficiently differentiable.

(c) **(1 XP extra)** The trapezoidal rule works amazingly well when the integrand $f(x)$ is smooth and periodic with period $b - a$. Can you explain why?

Solution.

(a) Let us write $g(x) = f(a + xh)$, so that

$$\mathrm{TR} := \frac{f(a)}{2} + f(a+h) + f(a+2h) + \ldots + f(b-h) + \frac{f(b)}{2} = \frac{g(0)}{2} + g(1) + g(2) + \ldots + g(N-1) + \frac{g(N)}{2}.$$

Euler–Maclaurin tells us that we have

$$\int_0^N g(x)\mathrm{d}x = \mathrm{TR} - \sum_{n=1}^{M} \frac{B_{2n}}{(2n)!}(g^{(2n-1)}(N) - g^{(2n-1)}(0)) - R_{2M}$$

with

$$R_M = \frac{(-1)^{M-1}}{M!} \int_0^N B_M(x - \lfloor x \rfloor) g^{(M)}(x)\mathrm{d}x.$$

Observe that $g^{(n)}(N) = h^n f^{(n)}(b)$ and $g^{(n)}(0) = h^n f^{(n)}(a)$. Moreover,

$$\int_0^N g(x)\mathrm{d}x = \int_0^N f(a + xh)\mathrm{d}x = \frac{1}{h}\int_a^b f(x)\mathrm{d}x.$$

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Hence, the error is

$$\int_a^b f(x)\mathrm{d}x - \mathrm{TR}\cdot h = -\sum_{n=1}^{M} \frac{B_{2n}h^{2n}}{(2n)!}(f^{(2n-1)}(b) - f^{(2n-1)}(a)) - h R_{2N}. \tag{1}$$

If $f \in C^2[a,b]$, then we can choose $M = 1$ and find that the error is

$$\int_a^b f(x)\mathrm{d}x - \mathrm{TR}\cdot h = -\frac{h^2}{12}(f'(b) - f'(a)) - h R_2 = -\frac{h^2}{12}(f'(b) - f'(a)) + O(h^2),$$

since

$$R_2 = -\frac{1}{2}\int_0^N B_2(x - \lfloor x \rfloor) g''(x)\mathrm{d}x = -\frac{h^2}{2}\int_0^N B_2(x - \lfloor x \rfloor) f''(a + x h)\mathrm{d}x = -\frac{h}{2}\int_a^b B_2(\ldots) f''(y)\mathrm{d}y = O(h),$$

where we used $g''(x) = h^2 f''(a + x h)$ as well as the fact that $B_2(x)$ is bounded on $[0,1]$.

(b) The first two terms of (1) are

$$-\frac{h^2}{12}(f'(b) - f'(a)) + \frac{h^4}{720}(f'''(b) - f'''(a)) + O(h^6).$$

That the error term is indeed of the form $O(h^6)$ is using the assumption that $f \in C^6[a,b]$.

(c) If $f(x)$ is smooth and periodic with period $b - a$, then we have $f^{(n)}(b) = f^{(n)}(a)$, and all the terms in (1), with the exception of the remainder term, are zero. Since f is smooth, we can choose M as large as we want to see that

$$\int_a^b f(x)\mathrm{d}x - \mathrm{TR}\cdot h = O(h^m)$$

for any $m > 0$. □