Problems #1

Special Functions & WZ Theory

AARMS Summer School, Dalhousie University
Jul 11 – Aug 5, 2016

First day warmup problems

Problem 1. (2 XP)

(a) We wish to find the ordinary generating function $G(x)$ of the Fibonacci sequence F_n. We sum the recurrence relation $F_n = F_{n-1} + F_{n-2}$ over n to get

$$G(x) = \sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_{n-1} x^n + \sum_{n=0}^{\infty} F_{n-2} x^n = x G(x) + x^2 G(x),$$

which implies $(1 - x - x^2) G(x) = 0$. Correct this (obviously wrong) argument!

(b) The *Pell numbers* P_n are defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$. Find a closed formula for P_n.

(c) Define the polynomials $F_n(x)$ by $F_0(x) = 0$, $F_1(x) = 1$ and $F_n(x) = x F_{n-1}(x) + F_{n-2}(x)$. Find the generating function for $(F_n(x))_{n=0,1,2,\ldots}$. Find a closed formula for $F_n(x)$ and show that it specializes to the one for the Fibonacci numbers and the Pell numbers.

Problem 2. (3 XP) The *Lucas numbers* L_n are the numbers defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$.

(a) Determine the ordinary generating function for the Lucas numbers.

(b) Let V be the set of all complex sequences $(X_n)_{n=0,1,2,\ldots}$ satisfying $X_n = X_{n-1} + X_{n-2}$ for all $n \geqslant 2$. Show that V is a 2-dimensional vector space over $\mathbb{C}$. Conclude that the Fibonacci and Lucas numbers form a basis.

(c) Prove that $L_n = F_{n-1} + F_{n+1}$ and that $5F_n = L_{n-1} + L_{n+1}$.

(d) Prove that $L_n = F_{2n} / F_n$.

(e) Determine, if possible, the limit of L_n / F_n as $n \to \infty$.

Exploring using Sage

Problem 3. (1 XP) Use Sage to compute the, say, first ten Taylor coefficients of $x / (1 - x - x^2)$. Are they Fibonacci numbers?

Problem 4. (1 XP) Find a rational number continuing the pattern

$$0.0001000100020003000500080013\ldots$$

Then, use Sage to compute that number to 100 decimal digits for verification.

Sage challenge: Can you find a way to discover the rational number from just the given digits (not using any knowledge about Fibonacci numbers)?

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