Problems #1

First day warmup problems

Problem 1. (2 XP)

(a) We wish to find the ordinary generating function $G(x)$ of the Fibonacci sequence F_n. We sum the recurrence relation $F_n = F_{n-1} + F_{n-2}$ over n to get

$$G(x) = \sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_{n-1} x^n + \sum_{n=0}^{\infty} F_{n-2} x^n = xG(x) + x^2 G(x),$$

which implies $(1 - x - x^2)G(x) = 0$. Correct this (obviously wrong) argument!

(b) The *Pell numbers* P_n are defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$. Find a closed formula for P_n.

(c) Define the polynomials $F_n(x)$ by $F_0(x) = 0$, $F_1(x) = 1$ and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$. Find the generating function for $(F_n(x))_{n=0,1,2,\ldots}$. Find a closed formula for $F_n(x)$ and show that it specializes to the one for the Fibonacci numbers and the Pell numbers.

Solution.

(a) Note that two of the intermediate sums involve Fibonacci numbers with negative index. The wrong argument proceeds under the assumption that we can just let $F_{-1} = F_{-2} = 0$. This, however, is inconsistent with the recursion $F_n = F_{n-1} + F_{n-2}$. If $F_{-1} = F_{-2} = 0$, then the recursion, would imply $F_n = 0$ for all $n \geqslant 0$, so that the generating function $G(x)$ indeed would be zero.

One way to fix the issue is to use values for F_{-1} and F_{-2} which are compatible with the recursion. Using the recursion with $n = 1$, we find $F_{-1} = F_1 - F_0 = 1$ and $F_{-2} = F_0 - F_{-1} = -1$. Hence,

$$G(x) = \sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_{n-1} x^n + \sum_{n=0}^{\infty} F_{n-2} x^n = (1 + xG(x)) + (-1 + x + x^2 G(x)),$$

which implies $(1 - x - x^2)G(x) = x$ and, therefore, $G(x) = x/(1 - x - x^2)$. Which is correct.

Alternatively, avoiding the issue of thinking about negative indices, we can make sure to use the recursion only with $n \geqslant 2$, and to use the initial conditions $F_0 = 0$, $F_1 = 1$.

$$G(x) = \sum_{n=0}^{\infty} F_n x^n = F_0 + F_1 x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n = x + xG(x) + x^2 G(x),$$

which again produces the correct known generating function.

(b) $P_n = \dfrac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$

$1 + \sqrt{2}$ is known as the “silver ratio”.

(c) The generating function satisfies the equation

$$G(z) = \sum_{n=0}^{\infty} F_n(x) z^n = F_0(x) + F_1(x) z + \sum_{n=2}^{\infty} xF_{n-1}(x) z^n + \sum_{n=2}^{\infty} F_{n-2}(x) z^n = z + xzG(z) + z^2 G(z),$$

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which we solve for

$$G(z) = \frac{z}{1 - xz - z^2}.$$

Partial fractions leads us to expressing $G(z)$ as

$$G(z) = \frac{1}{\alpha - \beta}\left[\frac{1}{1 - \alpha z} - \frac{1}{1 - \beta z}\right], \quad \alpha = \frac{x + \sqrt{x^2 + 4}}{2}, \quad \beta = \frac{x - \sqrt{x^2 + 4}}{2}.$$

Expanding the geometric series, we then find the explicit Binet-like formula

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In the case $x = 1$, this indeed reduces to the Binet formula for the Fibonacci numbers, and, in the case $x = 2$, this reduces to the Binet-like formula we derived earlier in this problem.

□

Problem 2. (3 XP) The *Lucas numbers* L_n are the numbers defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$.

(a) Determine the ordinary generating function for the Lucas numbers.

(b) Let V be the set of all complex sequences $(X_n)_{n=0,1,2,\ldots}$ satisfying $X_n = X_{n-1} + X_{n-2}$ for all $n \geqslant 2$. Show that V is a 2-dimensional vector space over $\mathbb{C}$. Conclude that the Fibonacci and Lucas numbers form a basis.

(c) Prove that $L_n = F_{n-1} + F_{n+1}$ and that $5F_n = L_{n-1} + L_{n+1}$.

(d) Prove that $L_n = F_{2n}/F_n$.

(e) Determine, if possible, the limit of L_n/F_n as $n \to \infty$.

Solution.

(a) $\dfrac{2 - x}{1 - x - x^2}$

(b) The first part of the claim is that V is a vector space. This means that, if a_n and b_n are two sequences satisfying the recurrence, then any linear combination $\alpha a_n + \beta b_n$ also satisfies the recurrence. This is certainly true, and holds for any homogeneous linear recurrence.

Secondly, we need to show that V is 2-dimensional. This follows from the fact that any sequence $(a_n)_{n \geqslant 0}$ satisfying the recursion is completely determined by the (initial) values a_0, a_1. More precisely, the map $V \to \mathbb{C}^2$ defined by $(a_n)_{n \geqslant 0} \mapsto (a_0, a_1)$ is an injective linear map. This map is also surjective, because the initial values can take any values, and so is an isomorphism.

To see that the Fibonacci and Lucas numbers form a basis, we only need to observe that, as vectors in the 2-dimensional space V, these two are linearly independent.

(c) Taking the generating function of both sides of $L_n = F_{n-1} + F_{n+1}$ shows that the equation is true, for all $n \geqslant 1$, if and only if

$$\sum_{n \geqslant 1} L_n x^n = \sum_{n \geqslant 1} F_{n-1} x^n + \sum_{n \geqslant 1} F_{n+1} x^n.$$

Using the known generating functions, this simplifies to

$$\frac{2 - x}{1 - x - x^2} - 2 = \frac{x^2}{1 - x - x^2} + \frac{1}{x}\frac{x}{1 - x - x^2} - 1,$$

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which is true, with both sides equalling $x(1+2x)/(1-x-x^2)$.

Along the same lines, we show that $5F_n = L_{n-1} + L_{n+1}$.

Here is an alternative approach which saves some calculations. Observe that, for both equations, each side is a solution to the linear recurrence $X_n = X_{n-1} + X_{n-2}$. Checking that the equations for two initial values therefore proves that the equations hold in general.

Thirdly, we could have used Binet's formula.

(d) This is straightforward to verify using Binet's formula for F_n and L_n.

(e) Likewise.

□

Exploring using Sage

Problem 3. (1 XP) Use Sage to compute the, say, first ten Taylor coefficients of $x/(1-x-x^2)$. Are they Fibonacci numbers?

Solution.

```
Sage] (x/(1-x-x^2)).series(x,11)
```

$$1\,x + 1\,x^2 + 2\,x^3 + 3\,x^4 + 5\,x^5 + 8\,x^6 + 13\,x^7 + 21\,x^8 + 34\,x^9 + 55\,x^{10} + \mathcal{O}(x^{11})$$

As expected, these are the Fibonacci numbers.
If we want to continue working with power series, the following computation in Sage is more suitable because it constructs an actual power series that can be manipulated as such (for instance, multiplied with other power series).

```
Sage] R.<x> = QQ[['x']]
Sage] R.set_default_prec(10)
Sage] x/(1-x-x^2)
```

$$x + x^2 + 2\,x^3 + 3\,x^4 + 5\,x^5 + 8\,x^6 + 13\,x^7 + 21\,x^8 + 34\,x^9 + 55\,x^{10} + O(x^{11})$$

□

Problem 4. (1 XP) Find a rational number continuing the pattern

$$0.0001000100020003000500080013\ldots$$

Then, use Sage to compute that number to 100 decimal digits for verification.

Sage challenge: Can you find a way to discover the rational number from just the given digits (not using any knowledge about Fibonacci numbers)?

Solution. Recall that

$$x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots = \sum_{n=1}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

Hence,

$$\left[\frac{x}{1-x-x^2}\right]_{x=\frac{1}{10^4}} = \frac{10000}{99989999}$$

is the rational number we are looking for.

```
Sage] rat = (x/(1-x-x^2)).subs(x=10^-4)
Sage] rat
```

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$$\frac{10000}{99989999}$$

```
Sage] rat.n(digits=100)
```

0.0001000100020003000500080013002100340055008901440233037706100987159725844181676609477713866163755\
5037141

One solution to the challenge is the following:

```
Sage] 0.00010001000200030005000800013.nearby_rational(max_denominator=10^10)
```

$$\frac{10000}{99989999}$$

The natural way to discover this possibility (that's what I did myself) is to declare, say, `num=0.33` and then explore available functions by typing `num.` (including the dot) and pressing `TAB`. □

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