**Example 108.** Under which conditions is

$$S(x) = \begin{cases} S_1(x), & \text{if } x \in [0,a], \\ S_2(x), & \text{if } x \in [a,b], \end{cases}$$

is a cubic spline? A natural cubic spline?

**Solution.** $S(x)$ is a cubic spline if $S_1(x)$ and $S_2(x)$ are cubic polynomials such that

$$S_1(a) = S_2(a), \quad S_1'(a) = S_2'(a), \quad S_1''(a) = S_2''(a).$$

$S(x)$ is a natural cubic spline if, in addition, $S_1''(0) = 0$ and $S_2''(b) = 0$.

**Comment.** Together with the three conditions coming from prescribing the values $S(0)$, $S(a)$ and $S(b)$, these are $8$ conditions in order for $S(x)$ to be a natural cubic spline. $8$ equations are just the right number to uniquely determine the underlying $2 \cdot 4 = 8$ unknowns.

**Example 109.** The following function $S(x)$ is a cubic spline.

$$S(x) = -1 - \frac{2}{9}(a-5)x - \frac{1}{3}(2a-1)x^2 + \frac{1}{36}x^3 \begin{cases} (-10a-13), & \text{if } x \in [-2,0], \\ 8(4a+7), & \text{if } x \in [0,1]. \end{cases}$$

(a) Spell out the conditions we need to check to see that this is a cubic spline.

(b) What are the underlying data points?

(c) Is there a choice of $a$ such that $S(x)$ is a natural cubic spline?

**Solution.**

(a) Write $S_1(x)$ for $S(x)$ on $[-2, 0]$ and $S_2(x)$ for $S(x)$ on $[0, 1]$. Then, as in the previous example, the conditions for $S(x)$ to be a cubic spline are

$$S_1(0) = S_2(0), \quad S_1'(0) = S_2'(0), \quad S_1''(0) = S_2''(0).$$

These conditions are visibly satisfied since the formulas for $S_1(x)$ and $S_2(x)$ agree up to a multiple of $x^3$.

(b) The knots of the spline are $-2, 0, 1$. We compute $S(-2) = 1$, $S(0) = -1$, $S(1) = 2$.
Hence the data points are $(-2, 1)$, $(0, -1)$, $(1, 2)$.

(c) In order for $S(x)$ to be a natural spline, we need $S''(-2) = 0$ as well as $S''(1) = 0$. Using

$$S''(x) = -\frac{2}{3}(2a-1) + \frac{1}{6}x \begin{cases} (-10a-13), & \text{if } x \in [-2,0], \\ 8(4a+7), & \text{if } x \in [0,1], \end{cases}$$

we have $S''(-2) = -\frac{2}{3}(2a-1) - \frac{1}{3}(-10a-13) = 2a+5$ and $S''(1) = -\frac{2}{3}(2a-1) + \frac{4}{3}(4a+7) = 4a+10$.
Both of these are $0$ if and only if $a = -\frac{5}{2}$. Therefore, $S(x)$ is a natural cubic spline if $a = -\frac{5}{2}$.

**Comment.** Can you explain why the two segments of the spline only differ in the cubic term?
[*Hint*: Note that $0$ is a knot and look again at the first part.]

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**Example 110.** Python Let us construct cubic splines using Python with `scipy`.

```
>>> from numpy import linspace
    from scipy import interpolate
```

**Comment.** Many basic functions like `linspace` are provided by both `numpy` and `scipy`.

We start by defining the data points that we wish to interpolate.

```
>>> xpoints = [1, 2, 4, 5, 7]
```

```
>>> ypoints = [2, 1, 4, 3, 2]
```

We can then construct the cubic spline with natural boundary conditions as follows.

```
>>> spline = interpolate.CubicSpline(xpoints, ypoints, bc_type='natural')
```

**Comment.** Other standard choices for the boundary conditions include `'not-a-knot'` (the default) as well as `'clamped'` and `'periodic'` (this one requires the first and last point to have the same $y$-coordinates).
https://docs.scipy.org/doc/scipy/reference/generated/scipy.interpolate.CubicSpline.html

The resulting natural cubic spline is piecewise defined by a collection of cubic polynomials. We can plot it as we did in Example 89 (this time we also include a legend).

```
>>> import matplotlib.pyplot as plt
```

```
>>> xplot = linspace(1, 7, 100)
```

```
>>> plt.plot(xplot, spline(xplot), '-', label='spline (natural)')
```

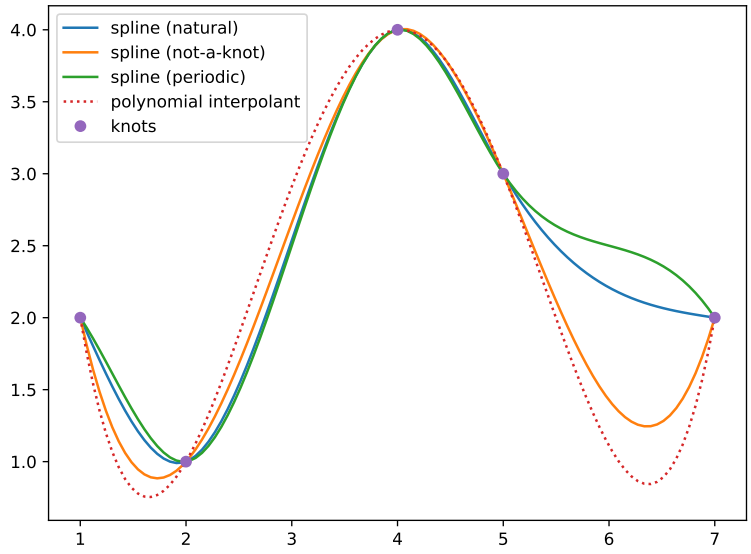
```
>>> plt.plot(xpoints, ypoints, 'o', label='knots')
```

```
>>> plt.legend()
```

```
>>> plt.show()
```

The resulting plot is a simpler version of the following one where we also included two other cubic splines as well as the polynomial interpolant:

**Homework.** Can you reproduce this plot?




Can you identify (some of) the splines without the labels? Try other knots and plot the splines!

**For instance.** The periodic spline is easily identified here because of the matching derivatives at the endpoints.
The natural spline is the one that is most like a clothesline pinned to the knots.
The not-a-knot spline is closer to polynomial interpolation.

If desired, we can access the piecewise polynomials as follows:

```
>>> spline.c

   [[ 6.37096774e-01 -6.49193548e-01  8.54838710e-01 -9.67741935e-02]
    [ 2.22044605e-16  1.91129032e+00 -1.98387097e+00  5.80645161e-01]
    [-1.63709677e+00  2.74193548e-01  1.29032258e-01 -1.27419355e+00]
    [ 2.00000000e+00  1.00000000e+00  4.00000000e+00  3.00000000e+00]]

>>>
```

For instance, the first column refers to $2 - 1.637(x-1) + 0.637(x-1)^3$ (the cubic used on $[1,2]$, the first interval) while the fourth column encodes $3 - 1.274(x-5) + 0.581(x-5)^2 - 0.097(x-5)^3$ (the cubic used on $[5,7]$, the last interval).

**Comment.** The exact cubics are $2 - \frac{203}{124}(x-1) + \frac{79}{124}(x-1)^3$ and $3 - \frac{79}{62}(x-5) + \frac{18}{31}(x-5)^2 - \frac{3}{31}(x-5)^3$.

Note how, for the first one, $S_1(x)$, we can immediately see that $S_1''(1) = 0$. Because we created a natural cubic spline, we also have $S_4''(7) = 0$. (Check it from the above exact formula!)

**Example 111.** In the case of four nodes/knots, how is the polynomial interpolant related to the cubic splines?

**Solution.** Note that the polynomial interpolant for four nodes is a cubic polynomial.

On the other hand, each cubic spline consists of three cubic polynomials $S_1, S_2, S_3$. In the case of the not-a-knot cubic spline, we have $S_1 = S_2$ as well as $S_3 = S_2$, which implies that all three are equal so that the not-a-knot cubic spline is a single cubic polynomial (interpolating the four given points).

Therefore, the polynomial interpolant must equal the not-a-knot cubic spline in this case.