

Newton's divided differences

Next, we observe an alternative way of computing the coefficients c_j in the Newton form

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \dots \quad (1)$$

for interpolating a function f at $x = x_0, x_1, x_2, \dots$

Example 83. Determine the first few coefficients. Below, we will use the following notation for these coefficients: $c_0 = f[x_0]$, $c_1 = f[x_0, x_1]$, $c_2 = f[x_0, x_1, x_2]$, $\dots$

Solution. For brevity, we write $y_j = f(x_j)$.

- Using (x_0, y_0) : $p(x_0) = c_0 \stackrel{!}{=} y_0$
 $f[x_0] = c_0 = y_0$

- Using (x_1, y_1) : $p(x_1) = c_0 + c_1(x_1 - x_0) \stackrel{!}{=} y_1$
 $f[x_0, x_1] = c_1 = \frac{y_1 - y_0}{x_1 - x_0}$

Note that the coefficient $f[x_0, x_1]$ is a **divided difference** (the slope of the line through the two points).

- Using (x_2, y_2) : $p(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) \stackrel{!}{=} y_2$
 $f[x_0, x_1, x_2] = c_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{y_2 - y_0}{x_2 - x_0} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_1} = \frac{f[x_0, x_2] - f[x_0, x_1]}{x_2 - x_1}$

The coefficient $f[x_0, x_1, x_2]$ is what we call a **divided difference of order 2**.

Definition 84. Define $f[x_0, x_1, \dots, x_n]$ to be the coefficient of x^n (the highest power of x) in the minimal-degree polynomial interpolating f at $x = x_0, x_1, \dots, x_n$.

Important. In other words, $f[x_0, x_1, \dots, x_n]$ is the coefficient c_n in the Newton form (1).

$f[x_0, x_1, \dots, x_n]$ is called a **divided difference of order n** of the function f because of the recursive relation illustrated in the previous example, which is proven in general in the next theorem.

Note that, by definition, $f[x_0, x_1, \dots, x_n]$ does not depend on the order of the points.

Theorem 85. The divided differences $f[x_0, x_1, \dots, x_n]$ are recursively determined by $f[a] = f(a)$ as well as the relation

$$f[P, a, b] = \frac{f[P, b] - f[P, a]}{b - a},$$

where P is a set of points.

For instance. With $P = x_1, \dots, x_{n-1}$ and $a = x_0$, $b = x_n$, the recursive relation becomes

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

Proof. Suppose that $P = \{x_0, \dots, x_n\}$ and that

$$p_0(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \dots + c_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

is the interpolating polynomial for $x_0, \dots, x_n$. Then

$$\begin{aligned} p_a(x) &= p_0(x) + f[P, a](x - x_0)(x - x_1) \cdots (x - x_n), \\ p_b(x) &= p_0(x) + f[P, b](x - x_0)(x - x_1) \cdots (x - x_n) \end{aligned}$$

are the interpolating polynomials for $x_0, \dots, x_n, a$ and $x_0, \dots, x_n, b$, respectively. Our goal is to determine the interpolating polynomial for $x_0, \dots, x_n, a, b$, which we can write as

$$p_{a,b}(x) = p_a(x) + C(x - x_0)(x - x_1) \cdots (x - x_n)(x - a)$$

where the constant $C = f[P, a, b]$ is to be determined. By construction, the polynomial already interpolates $x = x_0, \dots, x_n, a$. To ensure that it also interpolates $x = b$, we need

$$p_{a,b}(b) = p_a(b) + C(b - x_0)(b - x_1) \cdots (b - x_n)(b - a) \stackrel{!}{=} f(b).$$

Using $f(b) = p_b(b)$, we can now solve for C to find

$$C = \frac{p_b(b) - p_a(b)}{(b - x_0)(b - x_1) \cdots (b - x_n)(b - a)} = \frac{f[P, b] - f[P, a]}{b - a},$$

which is what we wanted to show. □

(Newton form using divided differences)

The Newton form of the polynomial $p(x)$ interpolating f at $x = x_0, x_1, \dots$ is

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \dots,$$

where the coefficients $c_n = f[x_0, x_1, \dots, x_n]$ can be computed using the triangular scheme:

	$f[\cdot]$	$f[\cdot, \cdot]$	$f[\cdot, \cdot, \cdot]$	$f[\cdot, \cdot, \cdot, \cdot]$	...
x_0	$f[x_0]$				
		$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$			
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$		
		$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dots$	
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$		...
		$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$		...	
x_3	$f[x_3]$		...		...

Note that the coefficients $c_n = f[x_0, x_1, \dots, x_n]$ needed for the Newton form appear at the top edge of the triangle (in the shaded cells).

Example 86. Determine the minimal polynomial interpolating the points $(-3, -1), (-1, 5), (0, 8), (2, -1)$.

Solution. (Newton, direct approach; again, for comparison) The interpolating polynomial in Newton form is

$$p(x) = c_0 + c_1(x+3) + c_2(x+3)(x+1) + c_3(x+3)(x+1)x.$$

We use the four points to solve for the coefficients c_i :

$$\begin{aligned} (-3, -1) &: c_0 = -1 \\ (-1, 5) &: \underbrace{c_0}_{-1} + 2c_1 = 5 \implies c_1 = 3 \\ (0, 8) &: \underbrace{c_0}_{-1} + \underbrace{3c_1}_3 + 3c_2 = 8 \implies c_2 = 0 \\ (2, -1) &: \underbrace{c_0}_{-1} + \underbrace{5c_1}_3 + \underbrace{15c_2}_0 + 30c_3 = -1 \implies c_3 = -\frac{1}{2} \end{aligned}$$

Hence, $p(x) = -1 + 3(x+3) - \frac{1}{2}(x+3)(x+1)x = -\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 8$.

Solution. (Newton, divided differences)

	$f[\cdot]$	$f[\cdot, \cdot]$	$f[\cdot, \cdot, \cdot]$	$f[\cdot, \cdot, \cdot, \cdot]$
-3	-1			
		$\frac{5 - (-1)}{-1 - (-3)} = 3$		
-1	5		$\frac{3 - 3}{0 - (-3)} = 0$	
		$\frac{8 - 5}{0 - (-1)} = 3$		$\frac{-\frac{5}{2} - 0}{2 - (-3)} = -\frac{1}{2}$
0	8		$\frac{-\frac{9}{2} - 3}{2 - (-1)} = -\frac{5}{2}$	
		$\frac{-1 - 8}{2 - 0} = -\frac{9}{2}$		
2	-1			

Accordingly, reading the coefficients from the top edge of the triangle (as shaded above), the Newton form is

$$p(x) = -1 + 3(x+3) - \frac{1}{2}(x+3)(x+1)x = -\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 8,$$

in agreement with what we had computed earlier.

Example 87. Determine the minimal polynomial interpolating $(0, -1), (2, 1), (3, 8)$.

Solution. (Lagrange) The interpolating polynomial in Lagrange form is:

$$\begin{aligned} p(x) &= -1 \frac{(x-2)(x-3)}{(0-2)(0-3)} + 1 \frac{(x-0)(x-3)}{(2-0)(2-3)} + 8 \frac{(x-0)(x-2)}{(3-0)(3-2)} \\ &= -\frac{1}{6}(x-2)(x-3) - \frac{1}{2}x(x-3) + \frac{8}{3}x(x-2) = 2x^2 - 3x - 1 \end{aligned}$$

Solution. (Newton, direct approach) The interpolating polynomial in Newton form is

$$p(x) = c_0 + c_1(x-0) + c_2(x-0)(x-2).$$

We use the three points to solve for the coefficients c_i :

- $(0, -1)$: $c_0 = -1$.
- $(2, 1)$: $\underset{-1}{c_0} + 2c_1 = 1$, so that $c_1 = 1$.
- $(3, 8)$: $\underset{-1}{c_0} + \underset{1}{3c_1} + 3c_2 = 8$, so that $c_2 = 2$.

Hence, $p(x) = -1 + 1(x-0) + 2(x-0)(x-2) = 2x^2 - 3x - 1$.

Solution. (Newton, divided differences)

$$\begin{array}{rcl} 0: & -1 & \\ & \frac{1 - (-1)}{2 - 0} = 1 & \\ 2: & 1 & \frac{7 - 1}{3 - 0} = 2 \\ & \frac{8 - 1}{3 - 2} = 7 & \\ 3: & 8 & \end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle, the Newton form is

$$p(x) = -1 + 1(x-0) + 2(x-0)(x-2) = 2x^2 - 3x - 1.$$

Example 88. (homework) Repeat the previous example with the additional point $(1, -2)$.

Solution. (Newton, divided differences) Notice how only the shaded entries are new.

$$\begin{array}{rcl} 0: & -1 & \\ & \frac{1 - (-1)}{2 - 0} = 1 & \\ 2: & 1 & \frac{7 - 1}{3 - 0} = 2 \\ & \frac{8 - 1}{3 - 2} = 7 & \frac{2 - 2}{1 - 0} = 0 \\ 3: & 8 & \frac{5 - 7}{1 - 2} = 2 \\ & \frac{-2 - 8}{1 - 3} = 5 & \\ 1: & -2 & \end{array}$$

Since the point $(1, -2)$ is on the graph of $2x^2 - 3x - 1$, we obtained the same final polynomial. If we had added a point not on the graph, then we would have found a degree 3 polynomial interpolating the total of four points.

Important comment. This is a considerable advantage for many practical purposes since often one does not know a priori how many interpolation points to use. (Instead of an exact “0” for the newest coefficient, we would typically be looking for small new coefficients before stopping.)