Midterm #1

MATH 436/565 — Numerical Analysis
Monday, Sept 29, 2025

Please print your name:

No notes, fancy calculators or tools of any kind are permitted. There are 28 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (5 points) Suppose we wish to approximate the function $f(x) = 3 - 4x\ln(x)$.

(a) What is the 2nd Taylor polynomial $p_2(x)$ of $f(x)$ at $x = 1$?

(b) Provide an upper bound for the error of approximating $f(x)$ by $p_2(x)$ on the interval $[1, 2]$.

Solution.

(a) $f'(x) = -4\ln(x) - 4$

$f''(x) = -\frac{4}{x}$

Hence, the 2nd Taylor polynomial of $f(x)$ at $x = 1$ is

$$p_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 3 - 4(x-1) - 2(x-1)^2.$$

Comment. There is typically no reason to expand this out since this approximation is intended to be used for x of the form $1 + \delta$, where δ is small (i.e. for x close to 1).

(b) Taylor's theorem implies that

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-1)^3$$

for some ξ between 1 and x.

We compute that $f'''(x) = \frac{4}{x^2}$. This function is decreasing on $(0, \infty)$ and so, in particular, on $[1, 2]$. Therefore, the maximum absolute value on $[1, 2]$ is taken at $x = 1$ or $x = 2$. Since $|f'''(1)| = 4$ and $|f'''(2)| = 1$, we conclude that $|f'''(\xi)| \leqslant 4$.

On the other hand, $|(x-1)^3| \leqslant 1^3 = 1$ for all $x \in [1, 2]$.

We therefore conclude that the error on $[1, 2]$ is bounded by

$$|f(x) - p_2(x)| = \left|\frac{f^{(3)}(\xi)}{3!}(x-1)^3\right| \leqslant \frac{4}{3!} \cdot 1^3 = \frac{2}{3}.$$

Problem 2. (2 points) Newton's method applied to $x^3 - 7$ is equivalent to fixed-point iteration of which function?

Solution. Newton's method applied to $f(x) = x^3 - 7$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 7}{3x^2} = \frac{2x}{3} + \frac{7}{3x^2}.$$

Armin Straub
straub@southalabama.edu

Problem 3. (5 points) Determine all fixed points of $f(x) = \dfrac{4}{x+3}$. For each fixed point x^* determine whether fixed-point iteration of $f(x)$ converges locally to x^*. If so, determine the exact order of convergence as well as the rate.

Solution. To find the fixed points x^*, we solve $\frac{4}{x+3} = x$. Solving $4 = x(x+3)$, we find $x^* = -4$ and $x^* = 1$.
$f'(x) = -\frac{4}{(x+3)^2}$

- $f'(-4) = -4$

  Since $|f'(-4)| > 1$, fixed-point iteration does not converge locally to -4.

- $f'(1) = -\frac{1}{4}$

  Since $|f'(1)| < 1$, fixed-point iteration converges locally to 1. The convergence is linear with rate $|f'(1)| = \frac{1}{4}$.

Problem 4. (2 points) We have learned about the Newton method, the bisection method, the regula falsi method, and the secant method. List those methods that are guaranteed to converge.

Solution. Among these methods, only the bisection method and the regula falsi method always converge. The secant method and the Newton method only converge locally (that is, if the initial approximation is close enough to the desired root).

Problem 5. (2 points) Suppose that x^* is a root of $f(x)$. When does Newton's method fail to locally converge to x^* with order of convergence at least 2?

Solution. If $f'(x^*) = 0$ (i.e. if x^* is a repeated root), then Newton's method does not converge quadratically.

Problem 6. (2 points) Give one advantage of the bisection method over the Newton method, as well as one advantage of the Newton method over the bisection method.

Solution. The bisection method is guaranteed to converge. On the other hand, the Newton method usually converges much faster (quadratic rather than linear).

Armin Straub
straub@southalabama.edu

Problem 7. (3 points) Represent -6.25 as a single precision floating-point number according to IEEE 754.

Solution. $-6.25 = -(110.01)_2 = -(1.1001)_2 \cdot 2^2 = \boxed{-}\,1.\underbrace{\boxed{1001}}_{\text{binary}} \cdot 2^2$

The exponent 2 gets stored as $2 + 127 = \boxed{1000{,}0001}$.

Overall, -6.25 is stored as $\boxed{1}\boxed{1000{,}0001}\boxed{1001{,}0000{,}0000{,}0000{,}0000{,}000}$.

Problem 8. (3 points) Express $15/7$ in base 2. If necessary, indicate which digits repeat.

Solution. Note that $15/7 = 2 + 1/7$ so that $15/7 = (10.\cdots)_2$ with $1/7$ to be accounted for.

- $2 \cdot 1/7 = \boxed{0} + 2/7$
- $2 \cdot 2/7 = \boxed{0} + 4/7$
- $2 \cdot 4/7 = \boxed{1} + 1/7$ and now things repeat…

Hence, $15/7 = (10.001\cdots)_2$ and the final three digits 001 repeat: $15/7 = (10.001001001\cdots)_2$

Problem 9. (2 points) Express -5 in binary using the two's complement representation with 6 bits.

Solution. Since $5 = (000101)_2$, -5 is represented by 111011 (invert all bits, then add 1).

Alternatively, note that $-5 = -2^5 + 27$ and $27 = (111011)_2$ to arrive at the same representation.

Problem 10. (2 points) Indicate two reasons why floating-point numbers are used rather than fixed-point numbers.

Solution.

- When using fixed-point numbers, scaling a number typically results in a loss of precision.

  [For instance, dividing a number by 2^r and then multiplying it with 2^r loses r digits of precision (in particular, this means that it is computationally dangerous to change units).]

- When using fixed-point numbers, the range of numbers is limited.

  [For instance, the largest number is on the order of 2^N where N is the number of bits used for the integer part. On the other hand, a floating-point number can be of the order of 2^{2^M} where M is the number of bits used for the exponent.]

Armin Straub
straub@southalabama.edu

(extra scratch paper)