## Newton's divided differences

Next, we observe an alternative way of computing the coefficients $c_j$ in the Newton form

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \ldots \tag{1}$$

for interpolating a function $f$ at $x = x_0, x_1, x_2, \ldots$.

**Example 83.** Determine the first few coefficients. Below, we will use the following notation for these coefficients: $c_0 = f[x_0]$, $c_1 = f[x_0, x_1]$, $c_2 = f[x_0, x_1, x_2]$, ....

**Solution.** For brevity, we write $y_j = f(x_j)$.

- Using $(x_0, y_0)$: $p(x_0) = c_0 \overset{!}{=} y_0$
  $f[x_0] = c_0 = y_0$
- Using $(x_1, y_1)$: $p(x_1) = c_0 + c_1(x_1 - x_0) \overset{!}{=} y_1$
  $f[x_0, x_1] = c_1 = \frac{y_1 - y_0}{x_1 - x_0}$
  Note that the coefficient $f[x_0, x_1]$ is a **divided difference** (the slope of the line through the two points).
- Using $(x_2, y_2)$: $p(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) \overset{!}{=} y_2$
  $f[x_0, x_1, x_2] = c_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{y_2 - y_0}{x_2 - x_0} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_1} = \frac{f[x_0, x_2] - f[x_0, x_1]}{x_2 - x_1}$
  The coefficient $f[x_0, x_1, x_2]$ is what we call a **divided difference of order** $\mathbf{2}$.

**Definition 84.** Define $f[x_0, x_1, \ldots, x_n]$ to be the coefficient of $x^n$ (the highest power of $x$) in the minimal-degree polynomial interpolating $f$ at $x = x_0, x_1, \ldots, x_n$.

**Important.** In other words, $f[x_0, x_1, \ldots, x_n]$ is the coefficient $c_n$ in the Newton form (1).
$f[x_0, x_1, \ldots, x_n]$ is called a **divided difference of order** $\boldsymbol{n}$ of the function $f$ because of the recursive relation illustrated in the previous example, which is proven in general in the next theorem.
Note that, by definition, $f[x_0, x_1, \ldots, x_n]$ does not depend on the order of the points.

**Theorem 85.** The divided differences $f[x_0, x_1, \ldots, x_n]$ are recursively determined by $f[a] = f(a)$ as well as the relation

$$f[P, a, b] = \frac{f[P, b] - f[P, a]}{b - a},$$

where $P$ is a set of points.

**For instance.** With $P = x_1, \ldots, x_{n-1}$ and $a = x_0$, $b = x_n$, the recursive relation becomes

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

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**Proof.** Suppose that $P = \{x_0, \ldots, x_n\}$ and that

$$p_0(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

is the interpolating polynomial for $x_0, \ldots, x_n$. Then

$$\begin{aligned} p_a(x) &= p_0(x) + f[P, a](x - x_0)(x - x_1)\cdots(x - x_n), \\ p_b(x) &= p_0(x) + f[P, b](x - x_0)(x - x_1)\cdots(x - x_n) \end{aligned}$$

are the interpolating polynomials for $x_0, \ldots, x_n, a$ and $x_0, \ldots, x_n, b$, respectively. Our goal is to determine the interpolating polynomial for $x_0, \ldots, x_n, a, b$, which we can write as

$$p_{a,b}(x) = p_a(x) + C(x - x_0)(x - x_1)\cdots(x - x_n)(x - a)$$

where the constant $C = f[P, a, b]$ is to be determined. By construction, the polynomial already interpolates $x = x_0, \ldots, x_n, a$. To ensure that it also interpolates $x = b$, we need

$$p_{a,b}(b) = p_a(b) + C(b - x_0)(b - x_1)\cdots(b - x_n)(b - a) \stackrel{!}{=} f(b).$$

Using $f(b) = p_b(b)$, we can now solve for $C$ to find

$$C = \frac{p_b(b) - p_a(b)}{(b - x_0)(b - x_1)\cdots(b - x_n)(b - a)} = \frac{f[P, b] - f[P, a]}{b - a},$$

which is what we wanted to show. $\square$

**(Newton form using divided differences)**

The Newton form of the polynomial $p(x)$ interpolating $f$ at $x = x_0, x_1, \ldots$ is

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \ldots,$$

where the coefficients $c_n = f[x_0, x_1, \ldots, x_n]$ can be computed using the triangular scheme:

| | $f[\cdot]$ | $f[\cdot,\cdot]$ | $f[\cdot,\cdot,\cdot]$ | $f[\cdot,\cdot,\cdot,\cdot]$ | $\ldots$ |
|---|---|---|---|---|---|
| $x_0$ | $f[x_0]$ | | | | |
| | | $f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$ | | | |
| $x_1$ | $f[x_1]$ | | $f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$ | | |
| | | $f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$ | | $f[x_0, x_1, x_2, x_3] = \ldots$ | |
| $x_2$ | $f[x_2]$ | | $f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$ | | $\ldots$ |
| | | $f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$ | | $\ldots$ | |
| $x_3$ | $f[x_3]$ | | $\ldots$ | | $\ldots$ |

Note that the coefficients $c_n = f[x_0, x_1, \ldots, x_n]$ needed for the Newton form appear at the top edge of the triangle (in the shaded cells).

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**Example 86.** Determine the minimal polynomial interpolating the points $(-3,-1), (-1,5), (0,8)$, $(2,-1)$.

**Solution. (Newton, direct approach; again, for comparison)** The interpolating polynomial in Newton form is

$$p(x) = c_0 + c_1(x+3) + c_2(x+3)(x+1) + c_3(x+3)(x+1)x.$$

We use the four points to solve for the coefficients $c_i$:

$$\begin{array}{rcl}
(-3,-1) & : & c_0 = -1 \\
(-1,5) & : & \underbrace{c_0}_{-1} + 2c_1 = 5 \quad \Longrightarrow \quad c_1 = 3 \\
(0,8) & : & \underbrace{c_0}_{-1} + 3\underbrace{c_1}_{3} + 3c_2 = 8 \quad \Longrightarrow \quad c_2 = 0 \\
(2,-1) & : & \underbrace{c_0}_{-1} + 5\underbrace{c_1}_{3} + 15\underbrace{c_2}_{0} + 30c_3 = -1 \quad \Longrightarrow \quad c_2 = -\dfrac{1}{2}
\end{array}$$

Hence, $p(x) = -1 + 3(x+3) - \frac{1}{2}(x+3)(x+1)x = -\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 8$.

**Solution. (Newton, divided differences)**

$$\begin{array}{r|lllll}
 & f[\cdot] & f[\cdot,\cdot] & f[\cdot,\cdot,\cdot] & f[\cdot,\cdot,\cdot,\cdot] \\
\hline
-3 & -1 & & & \\
 & & \dfrac{5-(-1)}{-1-(-3)} = 3 & & \\
-1 & 5 & & \dfrac{3-3}{0-(-3)} = 0 & \\
 & & \dfrac{8-5}{0-(-1)} = 3 & & \dfrac{-\frac{5}{2}-0}{2-(-3)} = -\dfrac{1}{2} \\
0 & 8 & & \dfrac{-\frac{9}{2}-3}{2-(-1)} = -\dfrac{5}{2} & \\
 & & \dfrac{-1-8}{2-0} = -\dfrac{9}{2} & & \\
2 & -1 & & &
\end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle (as shaded above), the Newton form is

$$p(x) = -1 + 3(x+3) - \frac{1}{2}(x+3)(x+1)x = -\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 8,$$

in agreement with what we had computed earlier.

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**Example 87.** Determine the minimal polynomial interpolating $(0,-1),(2,1),(3,8)$.

**Solution. (Lagrange)** The interpolating polynomial in Lagrange form is:

$$\begin{aligned} p(x) &= -1\frac{(x-2)(x-3)}{(0-2)(0-3)}+1\frac{(x-0)(x-3)}{(2-0)(2-3)}+8\frac{(x-0)(x-2)}{(3-0)(3-2)} \\ &= -\frac{1}{6}(x-2)(x-3)-\frac{1}{2}x(x-3)+\frac{8}{3}x(x-2)=2x^2-3x-1 \end{aligned}$$

**Solution. (Newton, direct approach)** The interpolating polynomial in Newton form is

$$p(x)=c_0+c_1(x-0)+c_2(x-0)(x-2).$$

We use the three points to solve for the coefficients $c_i$:

- $(0,-1)$: $c_0=-1$.
- $(2,1)$: $\underbrace{c_0}_{-1}+2c_1=1$, so that $c_1=1$.
- $(3,8)$: $\underbrace{c_0}_{-1}+3\underbrace{c_1}_{1}+3c_2=8$, so that $c_2=2$.

Hence, $p(x)=-1+1(x-0)+2(x-0)(x-2)=2x^2-3x-1$.

**Solution. (Newton, divided differences)**

$$\begin{array}{llll} 0: & -1 & & \\ & & \frac{1-(-1)}{2-0}=1 & \\ 2: & 1 & & \frac{7-1}{3-0}=2 \\ & & \frac{8-1}{3-2}=7 & \\ 3: & 8 & & \end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle, the Newton form is

$$p(x)=-1+1(x-0)+2(x-0)(x-2)=2x^2-3x-1.$$

**Example 88. (homework)** Repeat the previous example with the additional point $(1,-2)$.

**Solution. (Newton, divided differences)** Notice how only the shaded entries are new.

$$\begin{array}{lllll} 0: & -1 & & & \\ & & \frac{1-(-1)}{2-0}=1 & & \\ 2: & 1 & & \frac{7-1}{3-0}=2 & \\ & & \frac{8-1}{3-2}=7 & & \frac{2-2}{1-0}=0 \\ 3: & 8 & & \frac{5-7}{1-2}=2 & \\ & & \frac{-2-8}{1-3}=5 & & \\ 1: & -2 & & & \end{array}$$

Since the point $(1,-2)$ is on the graph of $2x^2-3x-1$, we obtained the same final polynomial. If we had added a point not on the graph, then we would have found a degree $3$ polynomial interpolating the total of four points.

**Important comment.** This is a considerable advantage for many practical purposes since often one does not know a priori how many interpolation points to use. (Instead of an exact "$0$" for the newest coefficient, we would typically be looking for small new coefficients before stopping.)

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