Midterm #2

MATH 436/565 — Numerical Analysis
Wednesday, Nov 15, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 40 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (8 points) Obtain an approximation for $f''(x)$ using the values $f(x-3h)$, $f(x)$, $f(x+2h)$ as follows: determine the polynomial interpolation corresponding to these values and then use its second derivative to approximate $f''(x)$. You do **not** need to determine the order of the approximation or the leading term of the error.

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Problem 2. (8 points) Use the trapezoidal rule to approximate $\int_1^3 \frac{1}{x}\,\mathrm{d}x = \log(3)$.

(a) Use $h = 1$ and $h = \frac{1}{2}$.

(b) Using Richardson extrapolation, combine the previous two approximations to obtain an approximation of higher order. (No need to simplify your answer.)

(c) The extrapolated approximation is equivalent to the outcome of which method applied with $h = \frac{1}{2}$?

Problem 3. (3 points) Recall that a cubic spline $S(x)$ through $(x_0, y_0), \ldots, (x_n, y_n)$ with $x_0 < x_1 < \ldots < x_n$ is piecewise defined by n cubic polynomials $S_1(x), \ldots, S_n(x)$ such that $S(x) = S_i(x)$ for $x \in [x_{i-1}, x_i]$.

Name **two** common boundary conditions of cubic splines and state their mathematical definition.

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Problem 4. (9 points) We have shown that $A(h) = \frac{1}{2h}[f(x+h) - f(x-h)]$ is an approximation of $f'(x)$ of order 2.

(a) Determine the leading term of the error.

(b) Apply Richardson extrapolation to $A(h)$ and $A(2h)$ to obtain an approximation of $f'(x)$ of higher order.

(c) Explain in a sentence why the resulting approximation is of order 4 (rather than 3).

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Problem 5. (8 points) Determine the minimal polynomial $P(x)$ interpolating $(-1,1),(3,2),(5,2)$.

(a) Write down the polynomial in Lagrange form.

(b) Write down the polynomial in Newton form.

(c) Suppose the above points lie on the graph of a smooth function $f(x)$. Write down an "explicit" formula for $f(x)-P(x)$, the error when using the interpolating polynomial to approximate $f(x)$.

Problem 6. (4 points) Suppose that $f(x)$ is a smooth function such that $|f^{(n)}(x)| \leqslant 2 \cdot n!$ for all $x \in [-1,1]$ and all $n \geqslant 0$. Suppose we approximate $f(x)$ on the interval $[-1,1]$ by a polynomial interpolation $P(x)$. How many Chebyshev nodes do we need to use in order to guarantee that the maximal error is at most 10^{-3}?

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(extra scratch paper)