Midterm #2

MATH 436/565 — Numerical Analysis
Wednesday, Nov 15, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 40 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (8 points) Obtain an approximation for $f''(x)$ using the values $f(x-3h)$, $f(x)$, $f(x+2h)$ as follows: determine the polynomial interpolation corresponding to these values and then use its second derivative to approximate $f''(x)$. You do **not** need to determine the order of the approximation or the leading term of the error.

Solution. We first compute the polynomial $p(t)$ that interpolates the three points $(x-3h, f(x-3h))$, $(x, f(x))$, $(x+2h, f(x+2h))$ using Newton's divided differences:

$$\begin{array}{r|lll} & f[\cdot] & f[\cdot,\cdot] & f[\cdot,\cdot,\cdot] \\ \hline x-3h & f(x-3h) & & \\ & & \dfrac{f(x)-f(x-3h)}{3h} =: c_1 & \\ x & f(x) & & \dfrac{3f(x+2h)-5f(x)+2f(x-3h)}{6h\cdot 5h} =: c_2 \\ & & \dfrac{f(x+2h)-f(x)}{2h} & \\ x+2h & f(x+2h) & & \end{array}$$

Hence, reading the coefficients from the top edge of the triangle, the interpolating polynomial is

$$p(t) = f(x) + c_1(t-x+3h) + c_2(t-x+3h)(t-x).$$

Since $p'(t) = c_1 + c_2(2t-2x+3h)$ and $p''(t) = 2c_2$, we have

$$p''(x) = 2c_2 = \frac{3f(x+2h)-5f(x)+2f(x-3h)}{15h^2}.$$

This is our approximation for $f''(x)$.

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Problem 2. (8 points) Use the trapezoidal rule to approximate $\int_1^3 \frac{1}{x}\,\mathrm{d}x = \log(3)$.

(a) Use $h=1$ and $h=\frac{1}{2}$.

(b) Using Richardson extrapolation, combine the previous two approximations to obtain an approximation of higher order. (No need to simplify your answer.)

(c) The extrapolated approximation is equivalent to the outcome of which method applied with $h=\frac{1}{2}$?

Solution. Let us write $f(x)=\frac{1}{x}$.

(a) $\int_1^3 f(x)\,\mathrm{d}x \approx \frac{h}{2}[f(1)+2f(2)+f(3)] = \frac{1}{2}\left[1+2\cdot\frac{1}{2}+\frac{1}{3}\right] = \frac{7}{6} \approx 1.1667$

$\int_1^3 f(x)\,\mathrm{d}x \approx \frac{h}{2}\left[f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)+f(3)\right] = \frac{1}{4}\left[1+2\cdot\frac{2}{3}+2\cdot\frac{1}{2}+2\cdot\frac{2}{5}+\frac{1}{3}\right]$
$=\frac{67}{60}\approx 1.1167$

(b) Let us write $A(h)$ and $A\left(\frac{h}{2}\right)$ for our two approximations, and A^* for the true value of the integral.

Since $A(h)$ is an approximation of order 2, we expect $A(h)\approx A^*+Ch^2$ for some constant C.

Correspondingly, $A\left(\frac{h}{2}\right)\approx A^*+\frac{1}{4}Ch^2$. Hence, $4A\left(\frac{h}{2}\right)-A(h)\approx(4-1)A^*=3A^*$.

Therefore, the Richardson extrapolation is $\frac{1}{3}\left[4A\left(\frac{h}{2}\right)-A(h)\right]=\frac{1}{3}\left[4\cdot\frac{67}{60}-\frac{7}{6}\right]=\frac{11}{10}=1.1$.

(c) Simpson's rule

Problem 3. (3 points) Recall that a cubic spline $S(x)$ through $(x_0,y_0),\ldots,(x_n,y_n)$ with $x_0<x_1<\ldots<x_n$ is piecewise defined by n cubic polynomials $S_1(x),\ldots,S_n(x)$ such that $S(x)=S_i(x)$ for $x\in[x_{i-1},x_i]$.

Name **two** common boundary conditions of cubic splines and state their mathematical definition.

Solution. The following are common choices for the boundary conditions of cubic splines:

- *natural*: $S_1''(x_0)=S_n''(x_n)=0$

  The resulting splines are simply called *natural cubic splines*.

- *not-a-knot*: $S_1'''(x_1)=S_2'''(x_1)$ and $S_n'''(x_{n-1})=S_{n-1}'''(x_{n-1})$

- *periodic*: $S_1'(x_0)=S_n'(x_n)$ and $S_1''(x_0)=S_n''(x_n)$ (only makes sense if $y_0=y_n$)

There are other common choices such *clamped cubic splines* for which the first derivatives at the endpoints are being set ("clamped") to user-specified values.

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Problem 4. (9 points) We have shown that $A(h) = \frac{1}{2h}[f(x+h) - f(x-h)]$ is an approximation of $f'(x)$ of order 2.

(a) Determine the leading term of the error.

(b) Apply Richardson extrapolation to $A(h)$ and $A(2h)$ to obtain an approximation of $f'(x)$ of higher order.

(c) Explain in a sentence why the resulting approximation is of order 4 (rather than 3).

Solution.

(a) Our goal is to compute C such that $A(h) = f'(x) + Ch^2 + O(h^3)$. By Taylor's theorem, we have (note that, because we will divide by h, we know from the beginning that we need to compute up to h^3 in the following)

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4), \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + O(h^4). \end{aligned}$$

Subtracting these, we find

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f'''(x) + O(h^4).$$

Hence, $A(h) = f'(x) + \frac{h^2}{6}f'''(x) + O(h^3)$.

(b) We just showed that $A(h) = f'(x) + Ch^2 + O(h^3)$ for some constant C (we even determined C but it doesn't matter here). Correspondingly, $A(2h) = f'(x) + 4Ch^2 + O(h^3)$. Hence, $4A(h) - A(2h) = (4-1)f'(x) + O(h^3)$.

The Richardson extrapolation of $A(h)$ and $A(2h)$ therefore is:

$$\begin{aligned} \frac{4A(h) - A(2h)}{3} &= \frac{4}{3 \cdot 2h}[f(x+h) - f(x-h)] - \frac{1}{3 \cdot 4h}[f(x+2h) - f(x-2h)] \\ &= \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)] \end{aligned}$$

This is an approximation of $f'(x)$ of higher order.

Comment. With some more work, we find that the error is $-\frac{1}{30}f^{(5)}(x)h^4 + O(h^6)$ so that this is an approximation of order 4.

(c) In short, this is because our approximation $A(h)$ is an even function of h.

Because we started with an approximation of order 2, the Richardson extrapolation of $A(h)$ and $A(3h)$ has at least order 3. However, $A(h)$ is an even function of h (because $A(-h) = A(h)$). Consequently, $A(3h)$ as well as the extrapolation are even functions of h as well. Therefore, the error, which we know is of the form $Ch^3 + Dh^4 + O(h^5)$, can only feature even powers of h. Thus $C = 0$ and the error must be of order at least 4.

Problem 5. (8 points) Determine the minimal polynomial $P(x)$ interpolating $(-1,1),(3,2),(5,2)$.

(a) Write down the polynomial in Lagrange form.

(b) Write down the polynomial in Newton form.

(c) Suppose the above points lie on the graph of a smooth function $f(x)$. Write down an "explicit" formula for $f(x)-P(x)$, the error when using the interpolating polynomial to approximate $f(x)$.

Solution.

(a) The interpolating polynomial in Lagrange form is

$$P(x)=1\frac{(x-3)(x-5)}{(-1-3)(-1-5)}+2\frac{(x+1)(x-5)}{(3+1)(3-5)}+2\frac{(x+1)(x-3)}{(5+1)(5-3)}.$$

(If we had a reason to do so (we don't!), we could expand that expression to find $P(x)=\frac{11}{8}+\frac{x}{3}-\frac{x^2}{24}$.)

(b) Newton's divided differences for the four points are:

$$\begin{array}{r|llll}
-1 & 1 & & \\
 & & \frac{2-1}{3-(-1)}=\frac{1}{4} & \\
3 & 2 & & \frac{0-\frac{1}{4}}{5-(-1)}=-\frac{1}{24} \\
 & & \frac{2-2}{5-3}=0. & \\
5 & 2 & &
\end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle (as shaded above), the Newton form is

$$P(x)=1+\frac{1}{4}(x+1)-\frac{1}{24}(x+1)(x-3).$$

(Since the interpolating polynomial is unique, this polynomial must be the same as the one in the first part.)

(c) $f(x)-P(x)=\dfrac{f'''(\xi)}{3!}(x+1)(x-3)(x-5)$ for some $\xi\in[-1,5]$ (assuming that $x\in[-1,5]$ as well).

Problem 6. (4 points) Suppose that $f(x)$ is a smooth function such that $|f^{(n)}(x)|\leqslant 2\cdot n!$ for all $x\in[-1,1]$ and all $n\geqslant 0$. Suppose we approximate $f(x)$ on the interval $[-1,1]$ by a polynomial interpolation $P(x)$. How many Chebyshev nodes do we need to use in order to guarantee that the maximal error is at most 10^{-3}?

Solution. We know that, using n Chebyshev nodes, the error is bounded as

$$\max_{x\in[-,1,1]}|f(x)-P_{n-1}(x)|\leqslant\frac{1}{2^{n-1}n!}\max_{\xi\in[-,1,1]}|f^{(n)}(\xi)|\leqslant\frac{1}{2^{n-2}}.$$

We need to choose n so that $2^{n-2}\geqslant 10^3$.

Knowing that $2^{10}=1024>10^3$, we conclude that we need $n-2\geqslant 10$ and, thus, $n=12$ Chebyshev nodes.

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(extra scratch paper)

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