Midterm #2 – Practice

MATH 436/565 — Numerical Analysis
Midterm: Wednesday, Nov 15, 2023

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Determine the minimal polynomial $P(x)$ interpolating $(-2,1),(0,1),(1,1),(3,2)$.

(a) Write down the polynomial in Lagrange form.

(b) Write down the polynomial in Newton form.

(c) Suppose the above points lie on the graph of a smooth function $f(x)$. Write down an "explicit" formula for $f(x)-P(x)$, the error when using the interpolating polynomial to approximate $f(x)$.

Problem 2. Suppose we approximate $f(x)=\cos\left(\frac{x}{2}\right)$ by the polynomial $P(x)$ interpolating it at $x=1,2,3$.

(a) Without computing $P(x)$, give an upper bound for the error when $x=0$ and when $x=\frac{\pi}{2}$.

(b) For which x in $[0,\pi]$ is our bound for the error maximal? What is the bound in that case?

Problem 3. Suppose we approximate a function $f(x)$ by the polynomial $P(x)$ interpolating it at $x=-1,-\frac{2}{3},\frac{2}{3},1$. Suppose that we know that $|f^{(n)}(x)|\leqslant n$ for all $x\in[-1,1]$.

(a) Give an upper bound for the error when $x=-\frac{1}{6}$ and when $x=0$.

(b) Give an upper bound for the error for all $x\in[-1,1]$.

(c) Suppose we replace the nodes $-1,-\frac{2}{3},\frac{2}{3},1$ with four other values. For which choice of these four interpolation nodes is this upper bound for the error minimal?

(d) For this optimal choice, what is the upper bound for the error for all $x\in[-1,1]$?

Problem 4. Suppose that $f(x)$ is a smooth function such that $|f^{(n)}(x)|\leqslant n!$ for all $x\in[-1,1]$ and all $n\geqslant 0$. Suppose we approximate $f(x)$ on the interval $[-1,1]$ by a polynomial interpolation $P(x)$. How many Chebyshev nodes do we need to use in order to guarantee that the maximal error is at most 10^{-6}?

Problem 5. Determine the natural cubic spline through $(-3,1)$, $(0,3)$, $(2,1)$.

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Problem 6. Recall that a cubic spline $S(x)$ through $(x_0, y_0), \ldots, (x_n, y_n)$ with $x_0 < x_1 < \ldots < x_n$ is piecewise defined by n cubic polynomials $S_1(x), \ldots, S_n(x)$ such that $S(x) = S_i(x)$ for $x \in [x_{i-1}, x_i]$. Name three common boundary conditions of cubic splines and state their mathematical definition.

Problem 7. Obtain approximations for $f'(x)$ and $f''(x)$ using the values $f(x-2h)$, $f(x)$, $f(x+3h)$ as follows: determine the polynomial interpolation corresponding to these values and then use its derivatives to approximate those of f. In each case, determine the order of the approximation and the leading term of the error.

Problem 8. Suppose that $A\left(\frac{1}{4}\right) = a$ and $A\left(\frac{1}{10}\right) = b$ are approximations of order 4 of some quantity A^*. What is the approximation we obtain from using Richardson extrapolation?

Problem 9. We have shown that $A(h) = \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)]$ is an approximation of $f''(x)$ of order 2.

(a) Determine the leading term of the error.

(b) Apply Richardson extrapolation to $A(h)$ and $A(3h)$ to obtain an approximation of $f''(x)$ of higher order.

(c) Explain in a sentence why the resulting approximation is of order 4 (rather than 3).

Problem 10. Use the trapezoidal rule to approximate $\displaystyle\int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x = \frac{\pi}{4}$.

(a) Use $h = \frac{1}{3}$ and $h = \frac{1}{6}$.

(b) Using Richardson extrapolation, combine the previous two approximations to obtain an approximation of higher order. What are absolute and relative error?

(c) The extrapolated approximation is equivalent to the outcome of which method applied with $h = \frac{1}{6}$?

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