Midterm #1

MATH 436/565 — Numerical Analysis
Wednesday, Oct 4, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 35 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (5 points) Determine all fixed-points of $f(x) = \dfrac{4}{x+3}$. For each fixed-point x^* determine whether fixed-point iteration of $f(x)$ converges locally to x^*. If so, determine the exact order of convergence as well as the rate.

Solution. To find the fixed points x^*, we solve $\frac{4}{x+3} = x$. Solving $4 = x(x+3)$, we find $x^* = -4$ and $x^* = 1$.
$f'(x) = -\frac{4}{(x+3)^2}$

- $f'(-4) = -4$

  Since $|f'(-4)| > 1$, fixed-point iteration does not converge locally to -4.

- $f'(1) = -\frac{1}{4}$

  Since $|f'(1)| < 1$, fixed-point iteration converges locally to 1. The convergence is linear with rate $|f'(1)| = \frac{1}{4}$.

Problem 2. (2 points) We have learned about the Newton method, the bisection method, the regula falsi method, and the secant method. List those methods that are guaranteed to converge.

Solution. Among these methods, only the bisection method and the regula falsi method always converge. The secant method and the Newton method only converge locally (that is, if the initial approximation is close enough to the desired root).

Problem 3. (2 points) Describe briefly how the regula falsi method proceeds different from the bisection method.

Solution. The regula falsi method proceeds like the bisection method. However, instead of using the midpoint $\frac{a+b}{2}$ of the interval $[a, b]$, it uses the root of the secant line of $f(x)$ through $(a, f(a))$ and $(b, f(b))$.

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Problem 4. (3 points) Express $12/5$ in base 2. If necessary, indicate which digits repeat.

Solution. Note that $12/5 = 2 + 2/5$ so that $12/5 = (10.\cdots)_2$ with $2/5$ to be accounted for.

- $2 \cdot 2/5 = \boxed{0} + 4/5$
- $2 \cdot 4/5 = \boxed{1} + 3/5$
- $2 \cdot 3/5 = \boxed{1} + 1/5$
- $2 \cdot 1/5 = \boxed{0} + 2/5$ and now things repeat...

Hence, $12/5 = (10.0110\cdots)_2$ and the final four digits 0110 repeat: $12/5 = (10.01100110\cdots)_2$

Problem 5. (2 points) Express -17 in binary using the two's complement representation with 6 bits.

Solution. Since $17 = (010001)_2$, -17 is represented by 101111 (invert all bits, then add 1).

Alternatively, note that $-17 = -2^5 + 15$ and $15 = (1111)_2$ to arrive at the same representation.

Problem 6. (5 points) Consider $f(x) = (x + r)(x - 2)$ where r is some constant. Suppose we want to use Newton's method to calculate the root $x^* = 2$.

(a) For which values of r is Newton's method guaranteed to converge (at least) quadratically to $x^* = 2$?

(b) Analyze the case in which Newton's method does not converge quadratically to $x^* = 2$. Does it still converge? If so, determine the order and rate of convergence.

Solution. Recall that we showed that, if $f(x^*) = 0$ and $f'(x^*) \neq 0$, then Newton's method (locally) converges to x^* quadratically with rate $\frac{1}{2}|f''(x^*)/f'(x^*)|$.

(a) Newton's method is guaranteed to converge to 2 provided that $f'(2) \neq 0$.

Since $f(2) = 0$, we have $f'(2) = 0$ if and only if 2 is a repeated root which happens if and only if $r = -2$.

Alternatively, we could compute $f'(x) = 2x + r - 2$ so that $f'(2) = r + 2$. Thus $f'(2) = 0$ if and only if $r = -2$.

In either case, we conclude that Newton's method converges (at least) quadratically to $x^* = 2$ if $r \neq -2$.

(b) We need to analyze the case $r = -2$. In that case $f(x) = (x - 2)^2$.

It follows that $f'(x) = 2(x - 2)$.

Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(x-2)^2}{2(x-2)} = x - \frac{x-2}{2} = \frac{x}{2} + 1.$$

Hence, $g'(x) = \frac{1}{2}$ so that, in particular, $g'(2) = \frac{1}{2}$.

Since $0 \neq |g'(2)| < 1$ we conclude that Newton's method (locally) converges to $x^* = 2$. Moreover, the convergence is linear with rate $|g'(2)| = \frac{1}{2}$.

Problem 7. (2 points)

(a) Indicate one advantage of the bisection method over the Newton method.

(b) Indicate one advantage of the Newton method over the bisection method.

Solution.

(a) An advantage of the bisection method is that it is guaranteed to converge.

(b) An advantage of the Newton method is that, if it converges, it typically converges quadratically which is much faster than the essentially linear convergence of the bisection method. (Also, it does not require an initial interval that is guaranteed to contain a root.)

Problem 8. (2 points) Newton's method applied to $x^4 - 2$ is equivalent to fixed-point iteration of which function?

Solution. Newton's method applied to $f(x) = x^4 - 2$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^4 - 2}{4x^3} = \frac{3x}{4} + \frac{1}{2x^3}.$$

Problem 9. (4 points) We wish to compute the root $\sqrt{1/2}$ of $f(x) = 2x^2 - 1$ using the bisection method.

(a) Starting with the interval $[0, 1]$, apply two iterations of bisection. What is the resulting approximation of $\sqrt{1/2}$?

(b) After how many iterations can we guarantee that the absolute error is less than 0.001?

Solution.

(a) Note that $f(0) = -1 < 0$ while $f(1) = 1 > 0$. Hence, $f(x)$ must indeed have a root in the interval $[0, 1]$.

- The midpoint of $[0, 1]$ is $\frac{0+1}{2} = \frac{1}{2}$. Since $f\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{4} - 1 = -\frac{1}{2} < 0$, a root of $f(x)$ must be in $\left[\frac{1}{2}, 1\right]$.

- The midpoint of $\left[\frac{1}{2}, 1\right]$ is $\frac{1/2+1}{2} = \frac{3}{4}$. Since $f\left(\frac{3}{4}\right) = 2 \cdot \frac{9}{16} - 1 = \frac{1}{8} > 0$, a root of $f(x)$ must be in $\left[\frac{1}{2}, \frac{3}{4}\right]$.

The best approximation of $\sqrt{1/2} \approx 0.707$ at this point is the midpoint of the final interval: $\frac{5}{8} \approx 0.625$

(b) The width of the interval after n steps will be exactly $\ell = \frac{1-0}{2^n} = \frac{1}{2^n}$. Since $\sqrt{1/2}$ is contained in this interval, the absolute error of approximating it with the midpoint is at most $\ell/2 = \frac{1}{2^{n+1}}$. We need to select n so that $\frac{1}{2^{n+1}} < 10^{-3}$. Knowing that $2^{10} = 1024$ (and $\frac{1}{1024} < \frac{1}{1000}$), we conclude that we need 9 iterations.

Problem 10. (5 points) Suppose we wish to approximate the function $f(x) = 5 + 3x\ln(x)$.

(a) What is the 2nd Taylor polynomial $p_2(x)$ of $f(x)$ at $x = 1$?

(b) Provide an upper bound for the error of approximating $f(x)$ by $p_2(x)$ on the interval $[1, 2]$.

Solution.

(a) $f'(x) = 3\ln(x) + 3$

$f''(x) = \frac{3}{x}$

Hence, the 2nd Taylor polynomial of $f(x)$ at $x = 1$ is

$$p_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 5 + 3(x-1) + \frac{3}{2}(x-1)^2.$$

Comment. There is typically no reason to expand this out since this approximation is intended to be used for x of the form $1 + \delta$, where δ is small (i.e. for x close to 1).

(b) Taylor's theorem implies that

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-1)^3$$

for some ξ between 1 and x.

We compute that $f'''(x) = -\frac{3}{x^2}$. This function is increasing on $(0, \infty)$ and so, in particular, on $[1, 2]$. Therefore, the maximum absolute value on $[1, 2]$ is taken at $x = 1$ or $x = 2$. Since $|f'''(1)| = 3$ and $|f'''(2)| = \frac{3}{4}$, we conclude that $|f'''(\xi)| \leqslant 3$.

On the other hand, $|(x-1)^3| \leqslant 1^3 = 1$ for all $x \in [1, 2]$.

We therefore conclude that the error on $[1, 2]$ is bounded by

$$|f(x) - p_2(x)| = \left|\frac{f^{(3)}(\xi)}{3!}(x-1)^3\right| \leqslant \frac{3}{3!} \cdot 1^3 = \frac{1}{2}.$$

Comment. In this simple case, we can determine the maximal error exactly (without using Taylor's theorem). Since the function $f(x) - p_2(x)$ is decreasing on the interval $[1, 2]$, starting with the value 0, the maximal error must occur at $x = 2$ and is $\left|6\ln(2) - \frac{9}{2}\right| \approx 0.341$.

Problem 11. (3 points) Represent -4.5 as a single precision floating-point number according to IEEE 754.

Solution. $-4.5 = -(100.1)_2 = -(1.001)_2 \cdot 2^2 = \boxed{-}\,\underbrace{1.\boxed{001}}_{\text{binary}} \cdot 2^2$

The exponent 2 gets stored as $2 + 127 = \boxed{1000,0001}$.

Overall, -4.5 is stored as $\boxed{1}\boxed{1000,0001}\boxed{0010,0000,0000,0000,0000,000}$.

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(extra scratch paper)