Midterm #1 – Practice

MATH 436/565 — Numerical Analysis
Midterm: Wednesday, Oct 4, 2023

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Determine all values C such that fixed-point iteration of $f(x) = \frac{2x^2}{1+3x}$ converges locally to C. In each case, determine the exact order of convergence as well as the rate.

Problem 2. Consider $f(x) = (rx+1)(x^2-4)$ where r is some constant. Suppose we want to use Newton's method to calculate the root $x^* = 2$.

(a) For which values of r is Newton's method guaranteed to converge (at least) quadratically to $x^* = 2$?

(b) Analyze the cases in which Newton's method does not converge quadratically to $x^* = 2$. Does it still converge? If so, what can we say about the order and rate of convergence?

(c) For which values of r does Newton's method converge to $x^* = 2$ faster than quadratically?

Problem 3.

(a) Express 123 in base 5.

(b) Which number is represented by $(1101.011)_2$?

(c) Express $31/14$ in base 2. If necessary, indicate which digits repeat.

(d) Represent -6.5 as a single precision floating-point number according to IEEE 754.

(e) Represent 2^{-4} as a single precision floating-point number according to IEEE 754.

(f) Express -27 in binary using the two's complement representation with 6 bits.

Problem 4.

(a) What is IEEE 754? Describe two popular choices that it offers. How many bits are used for what purpose?

(b) Give two reasons why floating-point numbers are used rather than fixed-point numbers.

(c) Give an example of a situation where one should not use floating-point numbers for nonintegers. Offer an alternative.

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Problem 5. We wish to compute the root of $f(x) = x^3 - 3$.

(a) Starting with the interval $[1, 2]$, apply two iterations of the bisection method. What exactly does it provide? What is the final resulting approximation of $\sqrt[3]{3}$?

(b) After how many iterations can we guarantee that the absolute error is less than 0.001?

(c) Describe in words how the regula falsi method proceeds different from the bisection method.

(d) After n iterations of the regula falsi method, is it a good idea to use the midpoint of the final interval as an approximation of the root?

(e) How does the secant method relate to the regula falsi method?

(f) Give one advantage of the secant method over the regula falsi method, as well as one advantage of the regula falsi method over the secant method.

(g) Starting with the initial approximation 1, apply two iterations of the Newton method. Write down (but do not compute) the absolute and relative errors.

(h) Determine whether the Newton method converges locally to $\sqrt[3]{3}$. If so, determine the exact order and rate of convergence.

(i) Give one advantage of the Newton method over the secant method, as well as one advantage of the secant method over the Newton method.

Problem 6. Let $g(x) = \frac{3}{4}\left(x + \frac{1}{x^3}\right)$

(a) Determine the fixed points of $g(x)$.

(b) For each fixed point x^*, determine whether fixed-point iteration of $g(x)$ converges locally to x^*. If so, determine the exact order and rate of convergence.

(c) This fixed-point iteration was obtained by applying Newton's method to a function $f(x)$. Determine such a function $f(x)$.

Problem 7.

(a) We have learned about the bisection method, the regula falsi method, the secant method, and the Newton method. For each method, answer the following:

  (1) Does the method always converge?

  (2) What does the method require as input? What does it provide as output?

(b) True or false? If the Newton method converges, it must converge quadratically.

(c) Newton's method applied to $f(x) = \sin(x^2 + 1)$ is equivalent to fixed-point iteration of which function $g(x)$?

(d) Give a condition such that fixed-point iteration of a function $f(x)$ converges locally to some value C.

(e) Suppose that x^* is a root of $f(x)$. When does Newton's method fail to locally converge to x^* with order of convergence at least 2?

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Problem 8. Suppose we wish to approximate the function $f(x) = x^2 \ln(x)$.

(a) What is the 3rd Taylor polynomial $p_3(x)$ of $f(x)$ at $x = 1$?

(b) Provide an upper bound for the error of approximating $f(x)$ by $p_3(x)$ on the interval $[1, 3]$.

(c) Determine a value b such that the error of approximating $f(x)$ by $p_3(x)$ on the interval $[1, b]$ is less than 0.001.

Problem 9.

(a) Explain the following floating-point issue:

```
>>> 0.1 + 0.1 + 0.1 == 0.3
    False
>>> 0.1 + 0.1 + 0.1
    0.30000000000000004
```

(b) Explain the following floating-point issue:

```
>>> 10.**9 + 10.**-9 == 10.**9
    True
```

(c) Explain the following floating-point issue:

```
>>> def f(x):
        return (x-1)**99
>>> f(0.99) < 0
    True
>>> f(1.01) > 0
    True
>>> f(0.99) * f(1.01) < 0
    False
```

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