Example 149. Discretize the following Dirichlet problem:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 2 \\ u(x,2) &= 3 \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned} \quad \text{(BC)}$$

Use a step size of $h=\frac{1}{3}$.

Comment. Note that, for the Dirichlet problem as well as for our discretization, it doesn't matter that the boundary conditions aren't well-defined at the corners.

Solution. Note that our rectangle has side lengths 1 (in x direction) and 2 (in y direction). With a step size of $h=\frac{1}{3}$ we therefore get $4\cdot 7$ lattice points, namely the points

$$u_{m,n}=u(mh,nh), \quad m\in\{0,1,2,3\},\ n\in\{0,1,\ldots,6\}.$$

Further note that the boundary conditions determine the values of $u_{m,n}$ if $m=0$ or $m=3$ as well as if $n=0$ or $n=6$. This leaves $2\cdot 5=10$ points at which we need to determine the value of $u_{m,n}$.

Next, we approximate $u_{xx}+u_{yy}$ by $\frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)]$ (see previous example for how we obtained this finite difference approximation). Note that, if $u(x,y)=u_{m,n}$ is one of our lattice points, then the other four terms in the finite difference are lattice points as well; for instance, $u(x+h,y)=u_{m+1,n}$. The equation $u_{xx}+u_{yy}=0$ therefore translates into

$$u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}=0.$$

Spelling out these equation for each $m\in\{1,2\}$ and $n\in\{1,2,\ldots,5\}$, we get 10 (linear) equations for our 10 unknown values. For instance, here are the equations for $(m,n)=(1,1)$, $(1,2)$ as well as $(2,5)$:

$$\begin{aligned} u_{2,1}+\underbrace{u_{0,1}}_{=0}+u_{1,2}+\underbrace{u_{1,0}}_{=2}-4u_{1,1} &= 0 \\ u_{2,2}+\underbrace{u_{0,2}}_{=0}+u_{1,3}+u_{1,1}-4u_{1,2} &= 0 \\ &\vdots \\ \underbrace{u_{3,5}}_{=0}+u_{1,5}+\underbrace{u_{2,6}}_{=3}+u_{2,4}-4u_{2,5} &= 0 \end{aligned}$$

In matrix-vector form, these linear equations take the form:

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{1,4} \\ u_{1,5} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{2,4} \\ u_{2,5} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ \vdots \\ -3 \end{bmatrix}$$

Solving this system, we find $u_{1,1}\approx 0.7847$, $u_{1,2}\approx 0.3542$, ..., $u_{2,5}\approx 1.1597$.

For comparison, the corresponding exact values are $u\left(\frac{1}{3},\frac{1}{3}\right)\approx 0.7872$, $u\left(\frac{1}{3},\frac{2}{3}\right)\approx 0.3209$, ..., $u\left(\frac{2}{3},\frac{5}{3}\right)\approx 1.1679$.

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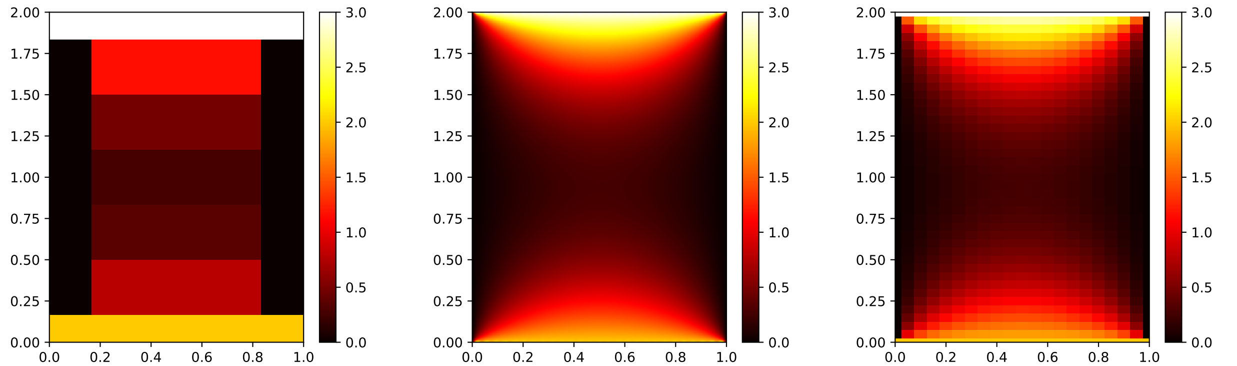
The three plots below visualize the discretized solution with $h = \frac{1}{3}$ from Example 149, the exact solution, as well as the discretized solution with $h = \frac{1}{20}$.



Comment. The first plot looks a bit overly rough because we chose not to interpolate the values. As we showed above, the approximate values at the ten lattice points are actually pretty decent for such a large step size.

Warning. The resulting linear systems quickly become very large. For instance, if we use a step size of $h = \frac{1}{100}$, then we need to determine roughly $100 \cdot 200 = 20{,}000$ ($99 \cdot 199$ to be exact) values $u_{m,n}$. The corresponding matrix M will have about $20{,}000^2 = 400{,}000{,}000$ entries, which is already challenging for a weak machine if we use generic linear algebra software. At this point it is important to realize that most entries of the matrix M are 0. Such matrices are called **sparse** and there are efficient algorithms for solving systems involving such matrices.

Example 150. Discretize the following Dirichlet problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 2 \\ u(x,1) &= 3 \qquad \text{(BC)} \\ u(0,y) &= 1 \\ u(2,y) &= 4 \end{aligned}$$

Use a step size of $h = \frac{1}{2}$.

Solution. Note that our rectangle has side lengths 2 (in x direction) and 1 (in y direction). With a step size of $h = \frac{1}{2}$ we therefore get $5 \cdot 3$ lattice points, namely the points

$$u_{m,n} = u(mh, nh), \quad m \in \{0,1,2,3,4\}, \ n \in \{0,1,2\}.$$

Further note that the boundary conditions determine the values of $u_{m,n}$ if $m = 0$ or $m = 4$ as well as if $n = 0$ or $n = 2$. This leaves $3 \cdot 1 = 3$ points at which we need to determine the value of $u_{m,n}$.

If we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx} + u_{yy} = 0$ translates into

$$u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0.$$

Spelling out these equation for each $m \in \{1,2,3\}$ and $n = 1$, we get 3 equations for our 3 unknown values:

$$\begin{aligned} u_{2,1} + \underbrace{u_{0,1}}_{=1} + \underbrace{u_{1,2}}_{=3} + \underbrace{u_{1,0}}_{=2} - 4u_{1,1} &= 0 \\ u_{3,1} + u_{1,1} + \underbrace{u_{2,2}}_{=3} + \underbrace{u_{2,0}}_{=2} - 4u_{2,1} &= 0 \\ \underbrace{u_{4,1}}_{=4} + u_{2,1} + \underbrace{u_{3,2}}_{=3} + \underbrace{u_{3,0}}_{=2} - 4u_{3,1} &= 0 \end{aligned}$$

In matrix-vector form, these linear equations take the form:

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \\ -9 \end{bmatrix}$$

Example 151. Consider the polygonal region with vertices $(0,0)$, $(4,0)$, $(4,2)$, $(2,2)$, $(2,3)$, $(0,3)$. We wish to find the steady-state temperature distribution $u(x,y)$ within this region if the temperature is A between $(0,0)$, $(4,0)$, and B elsewhere on the boundary.

Spell out the resulting equations when we discretize this problem using a step size of $h=1$.

Solution. As before, we write $u_{m,n}=u(mh,nh)$. Make a sketch!

$$\begin{array}{ccccc} & B & & & \\ B & u_{1,2} & B & B & \\ B & u_{1,1} & u_{2,1} & u_{3,1} & B \\ & A & A & A & \end{array}$$

If we approximate $u_{xx}+u_{yy}$ by $\frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx}+u_{yy}=0$ translates into

$$u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}=0.$$

Spelling out these equation in matrix-vector form, we obtain:

$$\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \end{bmatrix}=\begin{bmatrix} -A-B \\ -A-B \\ -A-2B \\ -3B \end{bmatrix}$$

Comment. Note that, because of the way we discretize, it matters that there is a well-defined temperature at the boundary vertex $(2,2)$. For the other vertices, we don't need a well-defined temperature (and so it is not a problem that it is unclear what the temperature should be at $(0,0)$ or $(4,0)$ where it jumps from A to B).

Fun in different bases

Example 152. Bases 2, 8 and 16 (binary, octal and hexadecimal) are commonly used in computer applications.

For instance, in JavaScript or Python, $0\mathrm{b}...$ means $(...)_2$, $0\mathrm{o}...$ means $(...)_8$, and $0\mathrm{x}...$ means $(...)_{16}$.

The digits $0,1,...,15$ in hexadecimal are typically written as $0,1,...,9,A,B,C,D,E,F$.

Example. FACE value in decimal? $(FACE)_{16}=15\cdot 16^3+10\cdot 16^2+12\cdot 16+14=64206$

Practical example. `chmod 664 file.tex` (change file permission)

664 are octal digits, consisting of three bits: $1=(001)_2$ execute (x), $2=(010)_2$ write (w), $4=(100)_2$ read (r)

Hence, 664 means rw,rw,r. What is rwx,rx,-? 750

By the way, a fourth (leading) digit can be specified (setting the flags: setuid, setgid, and sticky).

Example 153. (terrible jokes, parental guidance advised)

There are 10 types of people... those who understand binary, and those who don't.

Of course, you knew that. How about:

There are II types of people... those who understand Roman numerals, and those who don't.

It's not getting any better:

There are 10 types of people... those who understand hexadecimal, F the rest...

Example 154. (yet another joke) Why do mathematicians confuse Halloween and Christmas?

Because 31 Oct = 25 Dec.

Get it? $(31)_8=1+3\cdot 8=25$ equals $(25)_{10}=25$.

Fun borrowed from: https://en.wikipedia.org/wiki/Mathematical_joke

Excursion: A glance at numerical instability

Example 155. Recall that the quadratic equation $ax^2+bx+c=0$ has up to two solutions, which are given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Let us apply this formula to the example $(x-10^6)(x-10^{-10})=x^2-(10^6+10^{-10})x+10^{-4}$.

```
>>> b, c = -10**6-10**-10, 10**-4
```

```
>>> (-b + (b**2 - 4*c)**(1/2))/2
```

```
1000000.0
```

```
>>> (-b - (b**2 - 4*c)**(1/2))/2
```

```
1.1641532182693481e-10
```

Observe how the first root is computed correctly but the second root has a relative error of 0.164 (i.e. a percentage error of 16.4%). The reason for this large error is that the $-b$ and the $\sqrt{b^2-4ac}$ are very close to each other so that subtracting them results in a loss of precision.

In this example, we can avoid this loss of significant digits by observing that

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{b^2-(b^2-4ac)}{2a\left(-b\mp\sqrt{b^2-4ac}\right)}=\frac{-2c}{b\pm\sqrt{b^2-4ac}}.$$

Indeed, using the right-hand side instead, we get the following:

```
>>> -2*c / (b + (b**2-4*c)**(1/2))
```

```
858993.4592
```

```
>>> -2*c / (b - (b**2-4*c)**(1/2))
```

```
1e-10
```

Now, the first root has a large relative error (explain why!) but the second root is computed correctly. In the case $b<0$, we can therefore compute both roots without numerical issues by using $\frac{-b+\sqrt{b^2-4ac}}{2a}$ as well as $\frac{-2c}{b-\sqrt{b^2-4ac}}$ (instead of $\frac{-b-\sqrt{b^2-4ac}}{2a}$).

Comment. Make sure you see how this avoids in both cases the issue of subtracting numbers that are close to each other.