Example 106. Under which conditions is

$$S(x) = \begin{cases} S_1(x), & \text{if } x \in [0, a], \\ S_2(x), & \text{if } x \in [a, b], \end{cases}$$

is a cubic spline? A natural cubic spline?

Solution. $S(x)$ is a cubic spline if $S_1(x)$ and $S_2(x)$ are cubic polynomials such that

$$S_1(a) = S_2(a), \quad S_1'(a) = S_2'(a), \quad S_1''(a) = S_2''(a).$$

$S(x)$ is a natural cubic spline if, in addition, $S_1''(0) = 0$ and $S_2''(b) = 0$.

Comment. Together with the three conditions coming from prescribing the values $S(0)$, $S(a)$ and $S(b)$, these are 8 conditions in order for $S(x)$ to be a natural cubic spline. 8 equations are just the right number to uniquely determine the underlying $2 \cdot 4 = 8$ unknowns.

Example 107. The following function $S(x)$ is a cubic spline.

$$S(x) = -1 - \frac{2}{9}(a-5)x - \frac{1}{3}(2a-1)x^2 + \frac{1}{36}x^3 \begin{cases} (-10a-13), & \text{if } x \in [-2, 0], \\ 8(4a+7), & \text{if } x \in [0, 1]. \end{cases}$$

(a) Spell out the conditions we need to check to see that this is a cubic spline.

(b) What are the underlying data points?

(c) Is there a choice of a such that $S(x)$ is a natural cubic spline?

Solution.

(a) Write $S_1(x)$ for $S(x)$ on $[-2, 0]$ and $S_2(x)$ for $S(x)$ on $[0, 1]$. Then, as in the previous example, the conditions for $S(x)$ to be a cubic spline are

$$S_1(0) = S_2(0), \quad S_1'(0) = S_2'(0), \quad S_1''(0) = S_2''(0).$$

These conditions are visibly satisfied since the formulas for $S_1(x)$ and $S_2(x)$ agree up to a multiple of x^3.

(b) The knots of the spline are $-2, 0, 1$. We compute $S(-2) = 1$, $S(0) = -1$, $S(1) = 2$.
Hence the data points are $(-2, 1)$, $(0, -1)$, $(1, 2)$.

(c) In order for $S(x)$ to be a natural spline, we need $S''(-2) = 0$ as well as $S''(1) = 0$. Using

$$S''(x) = -\frac{2}{3}(2a-1) + \frac{1}{6}x \begin{cases} (-10a-13), & \text{if } x \in [-2, 0], \\ 8(4a+7), & \text{if } x \in [0, 1], \end{cases}$$

we have $S''(-2) = -\frac{2}{3}(2a-1) - \frac{1}{3}(-10a-13) = 2a+5$ and $S''(1) = -\frac{2}{3}(2a-1) + \frac{4}{3}(4a+7) = 4a+10$. Both of these are 0 if and only if $a = -\frac{5}{2}$. Therefore, $S(x)$ is a natural cubic spline if $a = -\frac{5}{2}$.

Comment. Can you explain why the two segments of the spline only differ in the cubic term?
[*Hint*: Note that 0 is a knot and look again at the first part.]

Example 108. `Python` Let us construct cubic splines using Python with `scipy`.

```
>>> from numpy import linspace
    from scipy import interpolate
```

Comment. Many basic functions like `linspace` are provided by both `numpy` and `scipy`.

We start by defining the data points that we wish to interpolate.

```
>>> xpoints = [1, 2, 4, 5, 7]
>>> ypoints = [2, 1, 4, 3, 2]
```

We can then construct the cubic spline with natural boundary conditions as follows.

```
>>> spline = interpolate.CubicSpline(xpoints, ypoints, bc_type='natural')
```

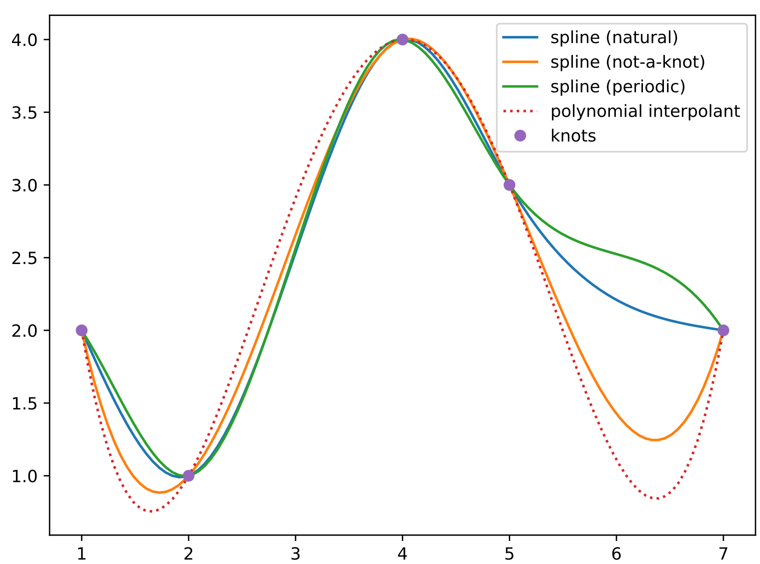
Comment. Other standard choices for the boundary conditions include `'not-a-knot'` (the default) as well as `'clamped'` and `'periodic'` (this one requires the first and last point to have the same y-coordinates).
https://docs.scipy.org/doc/scipy/reference/generated/scipy.interpolate.CubicSpline.html

The resulting natural cubic spline is piecewise defined by a collection of cubic polynomials. We can plot it as we did in Example 86 (this time we also include a legend).

```
>>> import matplotlib.pyplot as plt
>>> xplot = linspace(1, 7, 100)
>>> plt.plot(xplot, spline(xplot), '-', label='spline␣(natural)')
>>> plt.plot(xpoints, ypoints, 'o', label='knots')
>>> plt.legend()
>>> plt.show()
```

The resulting plot is a simpler version of the following one where we also included two other cubic splines as well as the polynomial interpolant:

Homework. Can you reproduce this plot?



Can you identify (some of) the splines without the labels? Try other knots and plot the splines!

For instance. The periodic spline is easily identified here because of the matching derivatives at the endpoints.
The natural spline is the one that is most like a clothesline pinned to the knots.
The not-a-knot spline is closer to polynomial interpolation.

If desired, we can access the piecewise polynomials as follows:

```
>>> spline.c

   [[ 6.37096774e-01 -6.49193548e-01  8.54838710e-01 -9.67741935e-02]
    [ 2.22044605e-16  1.91129032e+00 -1.98387097e+00  5.80645161e-01]
    [-1.63709677e+00  2.74193548e-01  1.29032258e-01 -1.27419355e+00]
    [ 2.00000000e+00  1.00000000e+00  4.00000000e+00  3.00000000e+00]]

>>>
```

For instance, the first column refers to $2 - 1.637(x - 1) + 0.637(x - 1)^3$ (the cubic used on $[1, 2]$, the first interval) while the fourth column encodes $3 - 1.274(x - 5) + 0.581(x - 5)^2 - 0.097(x - 5)^3$ (the cubic used on $[5, 7]$, the last interval).

Comment. The exact cubics are $2 - \frac{203}{124}(x - 1) + \frac{79}{124}(x - 1)^3$ and $3 - \frac{79}{62}(x - 5) + \frac{18}{31}(x - 5)^2 - \frac{3}{31}(x - 5)^3$. Note how, for the first one, $S_1(x)$, we can immediately see that $S_1''(1) = 0$. Because we created a natural cubic spline, we also have $S_4''(7) = 0$. (Check it from the above exact formula!)

Example 109. In the case of four nodes/knots, how is the polynomial interpolant related to the cubic splines?

Solution. Note that the polynomial interpolant for four nodes is a cubic polynomial.

On the other hand, each cubic spline consists of three cubic polynomials S_1, S_2, S_3. In the case of the not-a-knot cubic spline, we have $S_1 = S_2$ as well as $S_3 = S_2$, which implies that all three are equal so that the not-a-knot cubic spline is a single cubic polynomial (interpolating the four given points).

Therefore, the polynomial interpolant must equal the not-a-knot cubic spline in this case.