Example 87. Determine the minimal polynomial interpolating $(0,1),(1,2),(2,5)$.

Solution. (Lagrange, review) The interpolating polynomial in Lagrange form is:

$$\begin{aligned} p(x) &= 1\frac{(x-1)(x-2)}{(0-1)(0-2)} + 2\frac{(x-0)(x-2)}{(1-0)(1-2)} + 5\frac{(x-0)(x-1)}{(2-0)(2-1)} \\ &= \frac{1}{2}(x-1)(x-2) - 2x(x-2) + \frac{5}{2}x(x-1) \\ &= x^2+1 \end{aligned}$$

Solution. (Newton, divided differences)

$$\begin{array}{llll} 0: & 1 & & \\ & & \frac{2-1}{1-0}=1 & \\ 1: & 2 & & \frac{3-1}{2-0}=1 \\ & & \frac{5-2}{2-1}=3 & \\ 2: & 5 & & \end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle:

$$p(x) = 1 + 1(x-0) + 1(x-0)(x-1) = x^2+1$$

A mean value theorem for divided differences

Review. The **mean value theorem** (see Theorem 53; the special case $M=0$ of Taylor's theorem) states that, if $f(x)$ is differentiable, then

$$f[a,b] = \frac{f(b)-f(a)}{b-a} = f'(\xi)$$

for some ξ between a and b.

Recall that the Newton form of the polynomial interpolating $f(x)$ at $x = x_0, x_1, \ldots$ is

$$f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2) + \ldots$$

Note that this is somewhat similar to the Taylor expansion of $f(x)$ at $x=x_0$, which is

$$f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + \ldots$$

Indeed, if all the x_j are equal to x_0 (this is technically not allowed when interpolating, but you can still think of choosing them all close to x_0), then the Newton form would turn into a Taylor polynomial.
In that case, $f[x_0,x_1,\ldots,x_n]$ would become $\frac{1}{n!}f^{(n)}(x_0)$.

With that (as well as the mean value theorem and Taylor's theorem (see Theorem 52)) in mind, the next result does not come as a surprise.

Theorem 88. (mean value theorem for divided differences) If $f(x)$ is differentiable, then

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some ξ between the smallest and the largest of the x_i.

Proof. Without loss of generality, we may assume that $x_0 < x_1 < \cdots < x_n$ (because divided differences do not depend on the ordering of the points x_i).

Let $P(x)$ be the interpolation polynomial for f at $x_0, x_1, \ldots, x_n$. Then $d(x) = f(x) - P(x)$ has $n+1$ zeros, namely $x_0, x_1, \ldots, x_n$. The mean value theorem implies that between any two zeros of a function, there must be a zero of its derivative (this is often referred to as Rolle's theorem). It therefore follows that $d'(x)$ has n zeros (between x_0 and x_n). Applying the same argument to $d'(x)$, we then find that $d''(x)$ has $n-1$ zeros. Continuing like this, $d^{(n)}(x)$ must have a zero ξ between x_0 and x_n. As such,

$$0 = d^{(n)}(\xi) = f^{(n)}(\xi) - P^{(n)}(\xi).$$

Recall that $P(x)$ is a polynomial of degree n or less, and that its Newton form is

$$P(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

where $c_j = f[x_0, x_1, \ldots, x_j]$. Note that $P^{(n)}(x) = n! c_n = n! f[x_0, x_1, \ldots, x_n]$. We therefore conclude that

$$0 = d^{(n)}(\xi) = f^{(n)}(\xi) - P^{(n)}(\xi) = f^{(n)}(\xi) - n! f[x_0, x_1, \ldots, x_n],$$

which proves the claim. □

Comment. Note that this provides us with a way to numerically approximate an nth derivative $f^{(n)}(x)$. Namely, choose $n+1$ points $x_0, x_1, \ldots, x_n$ near x. Then $f^{(n)}(x) \approx \underbrace{n! f[x_0, x_1, \ldots, x_n]}_{= f^{(n)}(\xi)}$.

Bounding the interpolation error

Theorem 89. (interpolation error) Suppose that $f(x)$ is $n+1$ times continuously differentiable. Let $P_n(x)$ be the interpolating polynomial for $f(x)$ at $x_0, x_1, \ldots, x_n$. Then

$$f(x) - P_n(x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)}_{\text{interpolation error}}$$

for some ξ between the smallest and the largest of the x_i together with x.

Proof. Let $P_{n+1}(x)$ be the interpolating polynomial for $f(x)$ at $x_0, x_1, \ldots, x_n, x_{n+1}$. We know that

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + f[x_0, x_1, \ldots, x_{n+1}](x - x_0)(x - x_1)\cdots(x - x_n) \\ &= P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \end{aligned}$$

for some ξ between the smallest and the largest of the x_i together with x.

Given any fixed value t, choose $x_{n+1} = t$ in this formula (so that $P_{n+1}(t) = f(t)$) to conclude that

$$f(t) = P_{n+1}(t) = P_n(t) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - x_0)(t - x_1)\cdots(t - x_n),$$

which is the claimed expression for the error term (with x replaced by t). □

Example 90. Suppose we approximate $f(x)=\sin(x)$ by the polynomial $P(x)$ interpolating it at $x=0,\frac{\pi}{2},\pi$. Without computing $P(x)$, give an upper bound for the error when $x=\frac{\pi}{4}$.

[Compare with Example 86 where we computed and plotted $P(x)$.]

Solution. By Theorem 89, the error is

$$\sin(x)-P(x)=\frac{f^{(3)}(\xi)}{3!}(x-0)\left(x-\frac{\pi}{2}\right)(x-\pi),$$

where ξ is between 0 and π (provided that x is in $[0,\pi]$). Note that $f^{(3)}(x)=-\cos(x)$ so that $|f^{(3)}(\xi)|\leqslant 1$. Hence, the error is bounded by

$$|\sin(x)-P(x)|\leqslant\frac{1}{6}\left|x\left(x-\frac{\pi}{2}\right)(x-\pi)\right|.$$

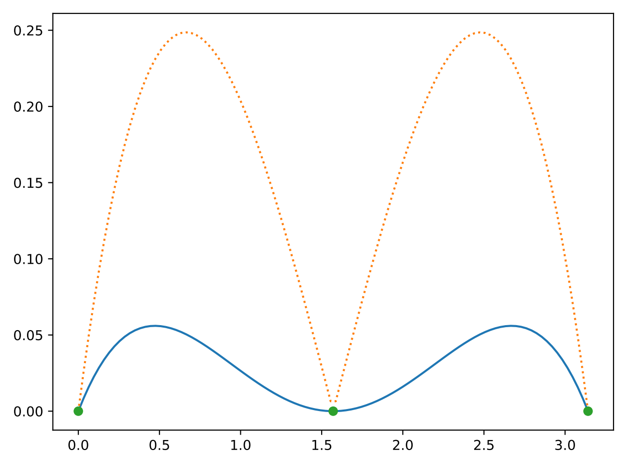
In particular, in the case $x=\frac{\pi}{4}$,

$$\left|\sin\left(\frac{\pi}{4}\right)-P\left(\frac{\pi}{4}\right)\right|\leqslant\frac{1}{6}\left|\frac{\pi}{4}\left(-\frac{\pi}{4}\right)\left(-\frac{3\pi}{4}\right)\right|=\frac{\pi^3}{128}\approx 0.242.$$

For comparison. In this particularly simple case, we can easily calculate the exact error.

Namely, since $\sin(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$ and $P(\frac{\pi}{4})=\frac{3}{4}$ (see Example 86), the actual error is $\left|\sin(\frac{\pi}{4})-P(\frac{\pi}{4})\right|\approx 0.0428$.

Below is a plot of the actual error (in blue) together with our bound (dotted).



Homework. Following what we did in Example 86, try to reproduce this plot.

For which x in $[0,\pi]$ is our bound for the error maximal? What is the bound in that case?

Solution. Recall that our bound for the error is $\frac{1}{6}\left|x\left(x-\frac{\pi}{2}\right)(x-\pi)\right|$.

$x\left(x-\frac{\pi}{2}\right)(x-\pi)$ is maximal on $[0,\pi]$ for $x=\left(1\pm\frac{1}{\sqrt{3}}\right)\frac{\pi}{2}\approx 0.664, 2.478$. (Fill in the details!)

The corresponding error bound is $\frac{1}{72\sqrt{3}}\pi^3\approx 0.249$.

Comment. Note that this shows that our earlier error bound for $x=\frac{\pi}{4}\approx 0.785$ was close to the worst case. That is not too much of a surprise since $\frac{\pi}{4}$ sits right between 0 and $\frac{\pi}{2}$ for which the error is 0 by construction.

For comparison. The actual maximal error occurs when $\cos(x)-\frac{4}{\pi}+\frac{8}{\pi^2}x=0$. (Why?!)

The approximate solutions are $x\approx 0.472, 2.670$ with corresponding (actual) error of 0.0560.

Make sure that you can identify both the x values and the error in the above plot.

Example 91. (homework) Suppose we approximate a function $f(x)$ by the polynomial $P(x)$ interpolating it at $x = -1, -\frac{1}{3}, \frac{1}{3}, 1$. Suppose that we know that $|f^{(n)}(x)| \leqslant n$ for all $x \in [-1, 1]$.

(a) Give an upper bound for the error when $x = -\frac{2}{3}$.

(b) Give an upper bound for the error when $x = 0$.

(c) Give an upper bound for the error for all $x \in [-1, 1]$.

Solution. By Theorem 89, the error is

$$f(x) - P(x) = \frac{f^{(4)}(\xi)}{4!}(x+1)\left(x+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)(x-1) = \frac{f^{(4)}(\xi)}{4!}(x^2-1)\left(x^2-\frac{1}{9}\right),$$

where ξ is between -1 and 1 (provided that $x \in [-1, 1]$). Since $\frac{1}{4!}|f^{(4)}(\xi)| \leqslant \frac{4}{4!} = \frac{1}{6}$, the error is bounded by

$$|f(x) - P(x)| \leqslant \frac{1}{6}\left|(x^2-1)\left(x^2-\frac{1}{9}\right)\right|.$$

(a) If $x = -\frac{2}{3}$, then this bound becomes $|f(x) - P(x)| \leqslant \frac{1}{6}\left|(x^2-1)\left(x^2-\frac{1}{9}\right)\right| = \frac{1}{6} \cdot \frac{5}{27} \approx 0.0309$.

(b) If $x = 0$, then this bound becomes $|f(x) - P(x)| \leqslant \frac{1}{6}\left|(x^2-1)\left(x^2-\frac{1}{9}\right)\right| = \frac{1}{6} \cdot \frac{1}{9} \approx 0.0185$.

Comment. It is not surprising that this error bound is better than the one for $x = -\frac{2}{3}$ since, roughly speaking, there are more interpolation nodes around 0.

(c) Consider $g(x) = (x^2-1)\left(x^2-\frac{1}{9}\right) = x^4 - \frac{10}{9}x^2 + \frac{1}{9}$. We need to compute $\max_{x\in[-1,1]} |g(x)|$.

Since $g(\pm 1) = 0$, the maximum value of $|g(x)|$ must be attained at a point where $g'(x) = 0$.

We compute $g'(x) = 4x^3 - \frac{20}{9}x$. Hence $g'(x) = 0$ if $x = 0$ or $x = \pm\frac{\sqrt{5}}{3}$.

Since $|g(0)| = \frac{1}{9}$ and $\left|g\left(\pm\frac{\sqrt{5}}{3}\right)\right| = \frac{16}{81} > \frac{1}{9}$, we conclude that $\max_{x\in[-1,1]} |g(x)| = \frac{16}{81}$.

Thus, our error bound is $\max_{x\in[-1,1]} |f(x) - P(x)| \leqslant \frac{1}{6}\max_{x\in[-1,1]}\left|(x^2-1)\left(x^2-\frac{1}{9}\right)\right| = \frac{1}{6} \cdot \frac{16}{81} \approx 0.0329$.

Example 92. Python We can approximate $\frac{1}{6}\max_{x\in[-1,1]}\left|(x^2-1)\left(x^2-\frac{1}{9}\right)\right| = \frac{1}{6} \cdot \frac{16}{81} \approx 0.0329$ as follows using 100 points.

```
>>> from numpy import linspace
>>> max([1/6*abs((x**2-1)*(x**2-1/9)) for x in linspace(-1,1,100)])
    0.0328984640831
```

Example 93. Python The following code measures how well a function f is approximated by the polynomial interpolating f at the given points. It returns an approximation of the maximal error on the interval $[a, b]$.

```
>>> from numpy import linspace, pi, cos, sin
>>> from scipy import interpolate
>>> def max_interpolation_error(f, a, b, xpoints, nr_sample_points):
        ypoints = [f(x) for x in xpoints]
        poly = interpolate.lagrange(xpoints, ypoints)
        max_error = max([abs(f(x)-poly(x)) for x in linspace(a,b,nr_sample_points)])
        return max_error
```

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Let us verify that this works using an example we have discussed before:

```
>>> max_interpolation_error(sin, 0, pi, [0,pi/2,pi], 100)

    0.0560067197786
```

This agrees with the maximal error that we observed at the end of Example 90. Let us look how the error develops as we add more points:

```
>>> [max_interpolation_error(sin, 0, pi, linspace(0,pi,n), 100) for n in range(2,9)]

    [0.99987412767387496, 0.056006719778558423, 0.043613266903306247,
    0.0018097268033398783, 0.0013114413108160916, 3.385907546618605e-05,
    2.4246231325325551e-05]
```

It is pleasing to see that the error decreases. However, as we will see in the next example, this does not have to be the case.

Comment. Note that the error seems to really decrease every second step (i.e. after adding two more points). Can you offer an explanation for what might be the cause of this?

Example 94. Python However, this is not the end of the story. It turns out that the interpolation error does not always go down if we add additional points.

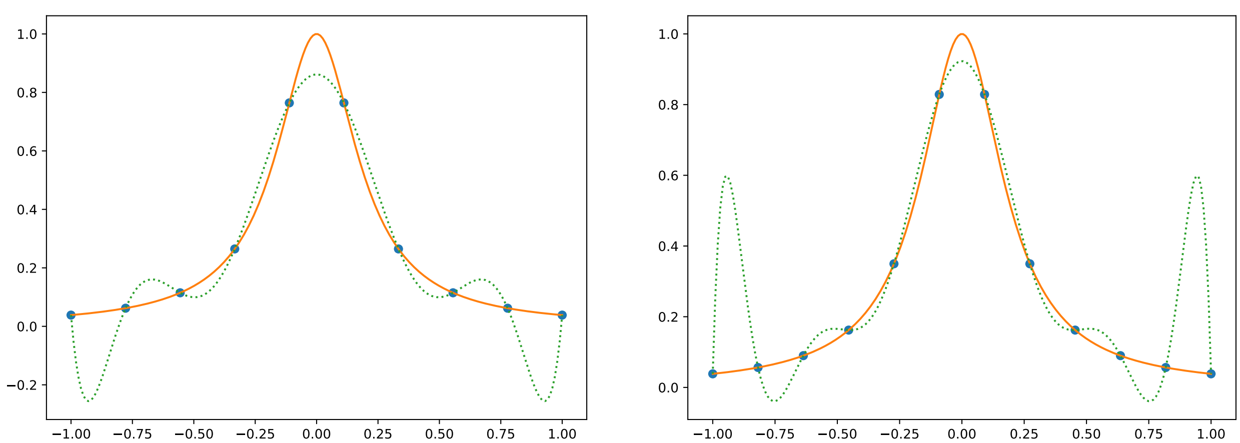
```
>>> def f(x):
        return 1/(1+25*x**2)

>>> [max_interpolation_error(f, -1, 1, linspace(-1,1,n), 100) for n in range(2,18)]

    [0.9589941912351845, 0.6459699748665507, 0.7044952736346626, 0.4382728746134098,
    0.43032461596244886, 0.6164015686420344, 0.24528527039305037, 1.0450782163781276,
    0.297971540151836, 1.9154342696798625, 0.5538529081557272, 3.6117015978042333,
    1.064460371610917, 7.189298472061041, 2.0967229089912, 14.013534491466531]
```

The function $f(x) = \frac{1}{1+25x^2}$ in this example is known as the **Runge function** and one can show that, by adding more points, the error grows without bound.

$$\lim_{n\to\infty} \max_{x\in[-,1,1]} |f(x) - P_n(x)| = \infty$$

https://en.wikipedia.org/wiki/Runge%27s_phenomenon

The following plots show the situation using 10 and 12 interpolation nodes.



While the approximation becomes better towards the center of the interval $[-1,1]$, the oscillations towards the ends of the interval become more violant (resulting in an increasing worst-case error). Next, we will see that we can avoid this issue if we don't choose equally spaced points but carefully chosen ones called **Chebyshev nodes**.