How computers represent functions

From classes like calculus, we are probably used to representing functions **symbolically**, such as:

$$f(x) = \frac{x\sin(3x-1)}{x^2+1}$$

Advantage. Because this is an exact expression, there is no loss of precision. Moreover, our calculus skills allow us to compute things like derivatives exactly.

Disadvantage. Computing with such formulas quickly results in very complicated expressions. Generally, things can get very slow very quickly. (Keep in mind the explosion in size of the rational numbers when we ran, for instance, Newton's method using exact numbers rather than floats.)

Moreover, in practice, a function of interest often simply doesn't have a symbolic expression. In such cases, we have to work with an approximation. There are many different ways to **numerically** represent and approximate a function.

For instance, we have already approximated functions using Taylor polynomials (these are particularly good at representing a function near a single point), which are truncations of the function's Taylor series. Another representation you have likely seen in other classes are Fourier series (combinations of sine and cosine functions) and their truncations.

A very basic numerical way to describe a function numerically is via a table of function values. In that case, how should we compute the function at an intermediate value not included in the table? This leads us to polynomial interpolation as well as to splines.

Interpolation

Suppose we have $d+1$ points $(x_0,y_0),(x_2,y_2),\ldots,(x_d,y_d)$. A function $f(x)$ is said to **interpolate** these points if $f(x_i)=y_i$ for all $i\in\{0,1,\ldots,d\}$.

An important case is to interpolate given points using a polynomial.

Why polynomials? Recall that any polynomial in x of degree n can be written as

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

with some coefficients a_j. The reason that these are so ubiquitous is that polynomials are precisely what can be constructed via addition and multiplication starting with numbers and a variable x.

Example 75. (warmup) Suppose we wish to interpolate the points $(0,1),(1,2),(2,5)$ using a polynomial.

Just as two points determine a line, these three points should determine a parabola $a_0+a_1x+a_2x^2$.

Indeed, note that there are three degrees of freedom (namely, a_0,a_1,a_2) and that each point gives rise to an equation. Three equations should determine the values of the three unknowns uniquely.

Let us spell out the three equations:

$(0,1)$: $a_0=1$

$(1,2)$: $a_0+a_1+a_2=2$

$(2,5)$: $a_0+2a_1+4a_2=5$

We can solve these to find $a_0=1$, $a_1=0$, $a_2=1$ so that the corresponding interpolating polynomial is x^2+1.

However, in general solving a system of equations is a lot of work! In the next sections, we will see that there are better and more efficient ways that avoid a lot of this work.

Also note that, from this point of view, it is not completely clear that the system of equations is always solvable.

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Example 76. Interpolate the points $(0,1),(1,2),(2,5)$ using a polynomial.

Solution. Without any computations, we can actually immediately write down such a polynomial:

$$p(x)=1\frac{(x-1)(x-2)}{(0-1)(0-2)}+2\frac{(x-0)(x-2)}{(1-0)(1-2)}+5\frac{(x-0)(x-1)}{(2-0)(2-1)}$$

This is the **Lagrange form** of the interpolating polynomial. Carefully look at this expression to see why it interpolates the three points! For instance, why is $p(1)=2$? Note that only the middle term contributes to $p(1)$ because the other two terms have a factor of $x-1$.

We can then simplify the above $p(x)$ to get $p(x)=\frac{1}{2}(x-1)(x-2)-2x(x-2)+\frac{5}{2}x(x-1)=x^2+1$.

As in the previous example, we can always write down an interpolating polynomial. In particular, this allows us to conclude the following.

Comment. Note that this is not surprising. It is a generalization of the well-known fact that two points determine a unique interpolating line.

Theorem 77. Given $d+1$ points $(x_0,y_0),(x_2,y_2),\ldots,(x_d,y_d)$ with distinct x_i, there exists a unique interpolating polynomial of degree at most d.

Proof. Existence follows because, as in the previous example, we can explicitly write down such an interpolating polynomial $p(x)$ as

$$p(x)=\sum_{j=0}^{d} y_j\frac{\prod_{i\neq j}(x-x_i)}{\prod_{i\neq j}(x_j-x_i)}.$$

For uniqueness, suppose there are two interpolating polynomials $p(x)$ and $q(x)$ of degree at most d. The difference $p(x)-q(x)$ has degree at most d. On the other hand, $p(x)-q(x)$ has $d+1$ roots because it vanishes for $x=x_0,x_1,\ldots,x_d$. Since a nonzero polynomial of degree at most d cannot have more than d roots, we conclude that $p(x)-q(x)=0$. □

Example 78. Interpolate the points $(0,1),(1,3),(2,5)$ using a polynomial of least degree.

Solution. The interpolating polynomial in Lagrange form is:

$$\begin{aligned}p(x) &= 1\frac{(x-1)(x-2)}{(0-1)(0-2)}+3\frac{(x-0)(x-2)}{(1-0)(1-2)}+5\frac{(x-0)(x-1)}{(2-0)(2-1)}\\ &= \frac{1}{2}(x-1)(x-2)-3x(x-2)+\frac{5}{2}x(x-1)\\ &= 2x+1\end{aligned}$$

Comment. Note that this is the special case where three points end up being on a single line. However, note that this wasn't obvious from the Lagrange form until we expanded it out to $2x+1$.

We will next see an alternative approach (Newton's divided differences) to obtaining the interpolating polynomial which, as one benefit (among others), will make it apparent that the resulting polynomial is of lower than expected degree.

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Polynomial interpolation: the Newton form

We have seen that, given $d+1$ points $(x_0, y_0), (x_2, y_2), \ldots, (x_d, y_d)$ with distinct x_i, there exists a unique interpolating polynomial $p(x)$ of degree at most d.

The **Lagrange form** of this polynomial is $p(x) = \sum_{j=0}^{d} y_j \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)}$.

The **Newton form** instead expresses the polynomial in the form

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \ldots$$

Comment. Note that the Newton form of a polynomial can be thought of as a generalization of Taylor expansion. Indeed, we get Taylor expansion around $x = c$ if we choose all x_i to be equal to c (of course, for the purposes of interpolation, the x_i will be different).

Example 79. Determine the minimal polynomial interpolating the points $(-3, -1), (-1, 5), (0, 8)$, $(2, -1)$.

Solution. (Lagrange) The interpolating polynomial in Lagrange form is:

$$\begin{aligned} p(x) &= -1\frac{(x+1)x(x-2)}{(-3+1)(-3)(-3-2)} + 5\frac{(x+3)x(x-2)}{(-1+3)(-1)(-1-2)} + 8\frac{(x+3)(x+1)(x-2)}{(0+3)(0+1)(0-2)} - 1\frac{(x+3)(x+1)x}{(2+3)(2+1)2} \\ &= -\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 8 \end{aligned}$$

Solution. (Newton, direct approach) The interpolating polynomial in Newton form is

$$p(x) = c_0 + c_1(x+3) + c_2(x+3)(x+1) + c_3(x+3)(x+1)x.$$

We use the four points to solve for the coefficients c_i:

$$\begin{aligned} (-3,-1) &: \; c_0 = -1 \\ (-1,5) &: \; \underbrace{c_0}_{-1} + 2c_1 = 5 \quad \Longrightarrow \quad c_1 = 3 \\ (0,8) &: \; \underbrace{c_0}_{-1} + 3\underbrace{c_1}_{3} + 3c_2 = 8 \quad \Longrightarrow \quad c_2 = 0 \\ (2,-1) &: \; \underbrace{c_0}_{-1} + 5\underbrace{c_1}_{3} + 15\underbrace{c_2}_{0} + 30c_3 = -1 \quad \Longrightarrow \quad c_2 = -\frac{1}{2} \end{aligned}$$

Hence, $p(x) = -1 + 3(x+3) - \frac{1}{2}(x+3)(x+1)x = -\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 8$.

Comment. Note how, by design of the Newton form, the equation for each point engaged one additional coefficient c_j, allowing us to solve for c_j (without having to combine several equations).

In particular, note how we are building intermediate interpolating polynomials:

- $c_0 + c_1(x+3) = -1 + 3(x+3)$ interpolates $(-3, 1), (-1, 6)$.
- $c_0 + c_1(x+3) + c_2(x+3)(x+1) = -1 + 3(x+3)$ interpolates $(-3, 1), (-1, 6), (0, 3)$.

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Newton's divided differences

Next, we observe an alternative way of computing the coefficients c_j in the Newton form

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \ldots \tag{1}$$

for interpolating a function f at $x = x_0, x_1, x_2, \ldots$.

Example 80. Determine the first few coefficients. Below, we will use the following notation for these coefficients: $c_0 = f[x_0]$, $c_1 = f[x_0, x_1]$, $c_2 = f[x_0, x_1, x_2]$, ...

Solution. For brevity, we write $y_j = f(x_j)$.

- Using (x_0, y_0): $p(x_0) = c_0 \stackrel{!}{=} y_0$
  $f[x_0] = c_0 = y_0$
- Using (x_1, y_1): $p(x_1) = c_0 + c_1(x_1 - x_0) \stackrel{!}{=} y_1$
  $f[x_0, x_1] = c_1 = \frac{y_1 - y_0}{x_1 - x_0}$
  Note that the coefficient $f[x_0, x_1]$ is a **divided difference** (the slope of the line through the two points).
- Using (x_2, y_2): $p(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) \stackrel{!}{=} y_2$
  $f[x_0, x_1, x_2] = c_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{y_2 - y_0}{x_2 - x_0} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_1} = \frac{f[x_0, x_2] - f[x_0, x_1]}{x_2 - x_1}$
  The coefficient $f[x_0, x_1, x_2]$ is what we call a **divided difference of order 2**.

Definition 81. Define $f[x_0, x_1, \ldots, x_n]$ to be the coefficient of x^n (the highest power of x) in the minimal-degree polynomial interpolating f at $x = x_0, x_1, \ldots, x_n$.

Important. In other words, $f[x_0, x_1, \ldots, x_n]$ is the coefficient c_n in the Newton form (1).

$f[x_0, x_1, \ldots, x_n]$ is called a **divided difference of order n** of the function f because of the recursive relation illustrated in the previous example, which is proven in general in the next theorem.

Note that, by definition, $f[x_0, x_1, \ldots, x_n]$ does not depend on the order of the points.

Theorem 82. The divided differences $f[x_0, x_1, \ldots, x_n]$ are recursively determined by $f[a] = f(a)$ as well as the relation

$$f[P, a, b] = \frac{f[P, b] - f[P, a]}{b - a},$$

where P is a set of points.

For instance. With $P = x_1, \ldots, x_{n-1}$ and $a = x_0$, $b = x_n$, the recursive relation becomes

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

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Proof. Suppose that $P = \{x_0, \ldots, x_n\}$ and that

$$p_0(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

is the interpolating polynomial for $x_0, \ldots, x_n$. Then

$$\begin{aligned} p_a(x) &= p_0(x) + f[P, a](x - x_0)(x - x_1)\cdots(x - x_{n-1})(x - x_n), \\ p_b(x) &= p_0(x) + f[P, b](x - x_0)(x - x_1)\cdots(x - x_{n-1})(x - x_n) \end{aligned}$$

are the interpolating polynomials for $x_0, \ldots, x_n, a$ and $x_0, \ldots, x_n, b$, respectively. The claim follows if we can show that

$$p_{a,b}(x) = p_a(x) + \frac{f[P, b] - f[P, a]}{b - a}(x - x_0)(x - x_1)\cdots(x - x_{n-1})(x - x_n)(x - a)$$

is the interpolating polynomial for $x_0, \ldots, x_n, a, b$. By construction, it interpolates $x = x_0, \ldots, x_n, a$. To see that it also interpolates $x = b$, note that

$$\begin{aligned} p_{a,b}(b) &= p_a(b) + (f[P, b] - f[P, a])\boxed{(b - x_0)(b - x_1)\cdots(b - x_{n-1})(b - x_n)} \\ &= p_0(b) + f[P, a]\boxed{(b - x_0)\cdots(b - x_{n-1})(b - x_n)} + (f[P, b] - f[P, a])\boxed{(b - x_0)\cdots(b - x_n)} \\ &= p_0(b) + f[P, b]\boxed{(b - x_0)\cdots(b - x_{n-1})(b - x_n)} \\ &= p_b(b) = f(b). \end{aligned}$$

□

(Newton form using divided differences)

The Newton form of the polynomial $p(x)$ interpolating f at $x = x_0, x_1, \ldots$ is

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \ldots,$$

where the coefficients $c_n = f[x_0, x_1, \ldots, x_n]$ can be computed using the triangular scheme:

	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$\ldots$
x_0	$f[x_0]$				
		$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$			
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$		
		$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \ldots$	
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$		$\ldots$
		$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$		$\ldots$	
x_3	$f[x_3]$		$\ldots$		$\ldots$

Note that the coefficients $c_n = f[x_0, x_1, \ldots, x_n]$ needed for the Newton form appear at the top edge of the triangle (in the shaded cells).

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Example 83. Determine the minimal polynomial interpolating the points $(-3,-1),(-1,5),(0,8),$ $(2,-1)$.

Solution. (Newton, direct approach; again, for comparison) The interpolating polynomial in Newton form is

$$p(x)=c_0+c_1(x+3)+c_2(x+3)(x+1)+c_3(x+3)(x+1)x.$$

We use the four points to solve for the coefficients c_i:

$$\begin{aligned}
(-3,-1) &: \ c_0=-1 \\
(-1,5) &: \ \underbrace{c_0}_{-1}+2c_1=5 \quad \Longrightarrow \quad c_1=3 \\
(0,8) &: \ \underbrace{c_0}_{-1}+3\underbrace{c_1}_{3}+3c_2=8 \quad \Longrightarrow \quad c_2=0 \\
(2,-1) &: \ \underbrace{c_0}_{-1}+5\underbrace{c_1}_{3}+15\underbrace{c_2}_{0}+30c_3=-1 \quad \Longrightarrow \quad c_2=-\frac{1}{2}
\end{aligned}$$

Hence, $p(x)=-1+3(x+3)-\frac{1}{2}(x+3)(x+1)x=-\frac{1}{2}x^3-2x^2+\frac{3}{2}x+8$.

Solution. (Newton, divided differences)

$$\begin{array}{r|lllll}
 & f[\cdot] & f[\cdot,\cdot] & f[\cdot,\cdot,\cdot] & f[\cdot,\cdot,\cdot,\cdot] \\
\hline
-3 & -1 & & & \\
 & & \frac{5-(-1)}{-1-(-3)}=3 & & \\
-1 & 5 & & \frac{3-3}{0-(-3)}=0 & \\
 & & \frac{8-5}{0-(-1)}=3 & & \frac{-\frac{5}{2}-0}{2-(-3)}=-\frac{1}{2} \\
0 & 8 & & \frac{-\frac{9}{2}-3}{2-(-1)}=-\frac{5}{2} & \\
 & & \frac{-1-8}{2-0}=-\frac{9}{2} & & \\
2 & -1 & & &
\end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle (as shaded above), the Newton form is

$$p(x)=-1+3(x+3)-\frac{1}{2}(x+3)(x+1)x=-\frac{1}{2}x^3-2x^2+\frac{3}{2}x+8,$$

in agreement with what we had computed earlier.