**Review.** If $f(x)$ is **analytic** around $x=c$, then it equals its **Taylor series** of $f(x)$ at $x=c$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \ldots$$

## Fixed-point iteration

**Definition 55.** $x^*$ is a **fixed point** of a function $f(x)$ if $f(x^*) = x^*$.

**Example 56.** Determine all fixed points of the function $f(x) = x^3$.

**Solution.** $x^3 = x$ has the three solutions $x^* = 0, \pm 1$ (and a cubic equation cannot have more than $3$ solutions). These are the fixed points.

**Idea.** Suppose $x^*$ is a fixed point of a continuous function $f$. If $x_n \approx x^*$, then $f(x_n) \approx f(x^*) = x^* \approx x_n$. If we can guarantee that $f(x_n)$ is closer to $x^*$ than $x_n$, then we can set

$$x_{n+1} = f(x_n),$$

with the expectation that iterating this process will bring us closer and closer to $x^*$.

**When does this converge?** This process converges if $|f(x_n) - x^*| < |x_n - x^*|$ for all $x_n$ close to $x^*$.

This condition is equivalent to $\left|\frac{f(x_n) - x^*}{x_n - x^*}\right| < 1$.

Since $x^* = f(x^*)$, we have $\frac{f(x_n) - x^*}{x_n - x^*} = \frac{f(x_n) - f(x^*)}{x_n - x^*} \approx f'(x^*)$ provided that $x_n$ is sufficiently close to $x^*$.

This essentially proves the following result. (See below for a full proof using the mean value theorem.)

**Theorem 57.** Suppose that $x^*$ is a fixed point of a continuously differentiable function $f$. If $|f'(x^*)| < 1$, then **fixed-point iteration**

$$x_{n+1} = f(x_n), \quad x_0 = \text{initial approximation},$$

converges to $x^*$ locally.

In that case, we say that $x^*$ is an **attracting fixed point**.

**Divergence.** If $|f'(x^*)| > 1$, then $x^*$ is a **repelling fixed point**. Our argument shows that fixed-point iteration will not converge to $x^*$ except in the "freak" case where $x_n \not\approx x^*$ but $f(x_n) = x^*$.

**Comment.** Local convergence means that we have convergence for all initial values $x_0$ close enough to $x^*$.

**Proof.** Note that

$$\begin{aligned} x_{n+1} - x^* &= f(x_n) - f(x^*) \\ &= f'(\xi_n)(x_n - x^*) \end{aligned}$$

where we applied the mean value theorem for the second equation and where $\xi_n$ is between $x_n$ and $x^*$. Thus

$$|x_{n+1} - x^*| = |f'(\xi_n)| \cdot |x_n - x^*|$$

Since $f'$ is continuous and $|f'(x^*)| < 1$, we have $|f'(x)| < \delta$ for some $\delta < 1$ for all $x$ sufficiently close to $x^*$. If $x_0$ is sufficiently to $x^*$ in that sense, then it follows that $|x_1 - x^*| < \delta \cdot |x_0 - x^*|$. In particular, $x_1$ is even closer to $x^*$ and we can repeat this argument to conclude that $|x_{n+1} - x^*| < \delta \cdot |x_n - x^*|$ for all $n$. This implies that $|x_n - x^*| < \delta^n \cdot |x_0 - x^*|$. Since $\delta < 1$, this further implies that $x_n$ converges to $x^*$. □


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**Example 58.** From a plot of $\cos(x)$, we can see that it has a unique fixed point $x^*$ in the interval $[0,1]$. Does fixed-point iteration converge locally to $x^*$?

**Solution.** If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. Since $|\sin(x)| < 1$ for all $x \in [0, 1]$, we conclude that $|f'(x^*)| < 1$. By Theorem 57, fixed-point iteration will therefore converge to $x^*$ locally.

**Comment.** We will continue this analysis in Example 66.

**Example 59.** Python Let us implement the fixed-point iteration of $\cos(x)$ from the previous example in Python.

```
>>> from math import cos
>>> def cos_iterate(x, n):
        for i in range(n):
            x = cos(x)
        return x
>>> [cos_iterate(1, n) for n in range(20)]
```

```
[1, 0.5403023058681398, 0.8575532158463934, 0.6542897904977791, 0.7934803587425656,
0.7013687736227565, 0.7639596829006542, 0.7221024250267077, 0.7504177617637605,
0.7314040424225098, 0.7442373549005569, 0.7356047404363474, 0.7414250866101092,
0.7375068905132428, 0.7401473355678757, 0.7383692041223232, 0.7395672022122561,
0.7387603198742113, 0.7393038923969059, 0.7389377567153445]
```

**For comparison.** The actual fixed point is $x^* \approx 0.7391$.

**Comment.** Instead of using a loop, we could also implement the above fixed-point iteration **recursively** in the following way (the recursive part is that the function is calling itself).

```
>>> def cos_iterate_recursively(x, n):
        if n > 0:
            return cos_iterate_recursively(cos(x), n-1)
        return x
>>> [cos_iterate_recursively(1, n) for n in range(20)]
```

```
[1, 0.5403023058681398, 0.8575532158463934, 0.6542897904977791, 0.7934803587425656,
0.7013687736227565, 0.7639596829006542, 0.7221024250267077, 0.7504177617637605,
0.7314040424225098, 0.7442373549005569, 0.7356047404363474, 0.7414250866101092,
0.7375068905132428, 0.7401473355678757, 0.7383692041223232, 0.7395672022122561,
0.7387603198742113, 0.7393038923969059, 0.7389377567153445]
```

Sometimes recursion results in cleaner code. However the use of loops is usually more efficient.

## Newton's method as a fixed-point iteration

Recall that Newton's method for finding a root of $f(x)$ proceeds from an initial approximation $x_0$ and iteratively computes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Note that this is equivalent to fixed-point iteration of the function $g(x) = x - \frac{f(x)}{f'(x)}$.

**Comment.** Note that $x^*$ is a fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$ if and only if $\frac{f(x^*)}{f'(x^*)} = 0$.

We have already proven a criterion for convergence of fixed-point iterations (Theorem 57). Our next goal is to develop the tools to analyze the speed of that convergence.

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**Example 60.** (We will continue this analysis in Example 67.)

(a) Newton's method applied to finding a root of $f(x) = x^3 - 2$ is equivalent to fixed-point iteration of which function $g(x)$?

(b) Determine whether Newton's method converges locally to $\sqrt[3]{2}$.

**Solution.**

(a) Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2}{3x^2} = \frac{2}{3}\left(x + \frac{1}{x^2}\right).$$

(b) By Theorem 57, Newton's method converges locally to $x^* = \sqrt[3]{2}$ if $|g'(x^*)| < 1$.

Since $g'(x) = \frac{2}{3} - \frac{4}{3x^3}$, we get $g'(x^*) = \frac{2}{3} - \frac{4}{3 \cdot 2} = 0$. Hence Newton's method converges locally to $\sqrt[3]{2}$.

**Important comment.** Notice that $g'(x^*) = 0$ is, in a way, the strongest sense in which $|g'(x^*)| < 1$. We will see shortly that $g'(x^*) = 0$ implies especially fast convergence of the type we observed in Example 45.

**Example 61. (homework)**

(a) What are the fixed points of $g(x) = \frac{x}{2} + \frac{1}{x}$?

(b) Does fixed-point iteration of $g(x)$ converge?

(c) Find a function $f(x)$ such that the fixed-point iteration of $g(x)$ is equivalent to Newton's method applied to $f(x)$.

(d) Inspired by the previous parts, suggest a fixed-point iteration to compute square roots.

**Solution.**

(a) Solving $\frac{x}{2} + \frac{1}{x} = x$, we find $x^2 = 2$ and thus $x = \pm\sqrt{2}$.

**Comment.** Note that $g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$. Suppose that $x < \sqrt{2}$. Then $2/x > \sqrt{2}$.

When iterating $g(x)$, we are averaging the underestimate and the overestimate, and it is reasonable to expect that the result is a better approximation.

(b) Since $g'(x) = \frac{1}{2} - \frac{1}{x^2}$, we have $g'(\pm\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0$. Hence, both fixed points are attracting fixed points. By Theorem 57, fixed-point iteration of $g(x)$ converges locally to both fixed points.

(c) We are looking for a function $f(x)$ such that $x - \frac{f(x)}{f'(x)} = g(x)$. Equivalently, $\frac{f'(x)}{f(x)} = \frac{1}{x - g(x)} = \frac{2x}{x^2 - 2}$.

This is a first-order differential equation which we can solve for $f(x)$ using separation of variables or by realizing that it is a linear DE. (Our approach below is equivalent to separation of variables.)

Note that $\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x))$. Thus, integrating both sides of the DE,

$$\ln(f(x)) = \int \frac{1}{x - g(x)}\mathrm{d}x = \int \frac{2x}{x^2 - 2}\mathrm{d}x = \ln|x^2 - 2| + C.$$

We conclude that fixed-point iteration of $g(x)$ is equivalent to Newton's method applied to $f(x) = x^2 - 2$.

**Comment.** The general solution of the DE has one degree of freedom (the $C$ above, which we chose as $0$). On the other hand, we know from the beginning that Newton's method applied to $f(x)$ and $Df(x)$ results in the same fixed-point iteration.

(d) Newton's method applied to $f(x) = x^2 - a$ is equivalent to fixed-point iteration of $g(x) = \frac{1}{2}(x + \frac{a}{x})$.

**Comment.** The resulting method for computing square roots $\sqrt{a}$ is known as the **Babylonian method**. It consists of starting with an approximation $x_0 \approx \sqrt{a}$ and then iteratively computing $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

https://en.wikipedia.org/wiki/Methods_of_computing_square_roots

## Order of convergence

**Example 62.** Suppose that $x_n$ converges to $x^*$ in such a way that the number of correct digits doubles from one term to the next. What does that mean in terms of the error $e_n = |x_n - x^*|$?

**Comment.** This is roughly what we observed numerically for the Newton method in Example 45.

**Comment.** It doesn't matter which base we are using because the number of digits in one base is a fixed constant multiple of the number of digits in another base. Make sure that this clear! (If unsure, how does the number of digits of an integer $x$ in base $2$ relate to the number of digits of $x$ in base $10$?)

**Solution.** Recall that the number of correct digits in base $b$ is about $-\log_b(e_n)$.
Doubling these from one term to the next means that $-\log_b(e_{n+1}) \approx -2\log_b(e_n)$.
Equivalently, $\log_b(e_{n+1}) - 2\log_b(e_n) = \log_b\left(\frac{e_{n+1}}{e_n^2}\right) \approx 0$.
This in turn is equivalent to $\frac{e_{n+1}}{e_n^2} \approx 1$.

**What if the number of correct digits triples?** By the above arguments, we would have $\frac{e_{n+1}}{e_n^3} \approx 1$.
Of course, there is nothing special about $2$ or $3$.

**Example 63.** Suppose that $x_n$ converges to $x^*$. Let $e_n = |x_n - x^*|$ be the error and $d_n = -\log_b(e_n)$ be the number of correct digits (in base $b$). If $d_{n+1} = A d_n + B$, what does that mean in terms of the error $e_n$?

**Solution.** $-\log_b(e_{n+1}) = -A\log_b(e_n) + B$ is equivalent to $\log_b(e_{n+1}) - A\log_b(e_n) = \log_b\left(\frac{e_{n+1}}{e_n^A}\right) = -B$.
This in turn is equivalent to $\frac{e_{n+1}}{e_n^A} = b^{-B}$.

This motivates the following definition.

**Definition 64.** Suppose that $x_n$ converges to $x^*$. Let $e_n = |x_n - x^*|$. We say that $x_n$ **converges to $x$ of order $q$ and rate $r$** if

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^q} = r.$$

**Order 1.** Convergence of order $1$ is called **linear convergence**. As in the previous example, the rate $r$ provides information on the number of additional correct digits per term.

**Order 2.** Convergence of order $2$ is also called **quadratic convergence**. As we saw above, it means that number of correct binary digits $d_n$ roughly doubles from one term to the next. More precisely, $d_{n+1} \approx 2d_n + B$ where the rate $r = 2^{-B}$ tells us that $B = -\log_2(r)$. [Note that $r$ has the advantage of being independent of the base in which we measure the number of correct digits.]

### Order of convergence of fixed-point iteration

**Theorem 65.** Suppose that $x^*$ is a fixed point of a sufficiently differentiable function $f$. Suppose that $|f'(x^*)| < 1$ so that, by Theorem 57, fixed-point iteration of $f(x)$ converges to $x^*$ locally. Then the convergence is of order $M$ with rate $\frac{1}{M!}|f^{(M)}(x^*)|$ where $M \geqslant 1$ is the smallest integer so that $f^{(M)}(x^*) \neq 0$.

**In particular.**

- If $f'(x^*) \neq 0$, then the convergence is linear with rate $|f'(x^*)|$.
- If $f'(x^*) = 0$ and $f''(x^*) \neq 0$, then the convergence is quadratic with rate $\frac{1}{2}|f''(x^*)|$.

**Comment.** Here, sufficiently differentiable means that $f$ needs to be $M$ times continuously differentiable so that we can apply Taylor's theorem.

**Proof.** By Taylor's theorem (Theorem 52), if $f'(x^*) = f''(x^*) = \cdots = f^{(M-1)}(x^*) = 0$ for some $M \geqslant 1$, then

$$f(x) = f(x^*) + \frac{1}{M!} f^{(M)}(\xi)(x - x^*)^M$$

for some $\xi$ between $x$ and $x^*$. We use this with $x$ replaced by $x_n$ to conclude that

$$x_{n+1} - x^* = f(x_n) - f(x^*) = \frac{1}{M!} f^{(M)}(\xi_n)(x_n - x^*)^M$$

for some $\xi_n$ between $x_n$ and $x^*$.
Thus

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^M} = \frac{1}{M!} f^{(M)}(\xi_n) \quad \underset{n\to\infty}{\longrightarrow} \quad \frac{1}{M!} f^{(M)}(x^*),$$

where the limit follows from the continuity of $f^{(M)}(x)$ (and convergence of $x_n \to x^*$). □

### Example 66. (cont'd) From a plot of $\cos(x)$, we can see that it has a unique fixed point in the interval $[0, 1]$. Does fixed-point iteration converge locally? If so, determine the order and the rate.

This is a continuation of Example 58.

**Solution.** If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. Since $|\sin(x)| < 1$ for all $x \in [0,\ 1]$, we conclude that $|f'(x^*)| < 1$. By Theorem 57, fixed-point iteration will therefore converge to $x^*$ locally.

Since $x^* \approx 0.7391$, we have $|f'(x^*)| \approx |\sin(0.7391)| \approx 0.6736$.

Because $f'(x^*) \neq 0$, we conclude that the order of convergence is $1$ and the rate is $0.6736$.

**Comment.** A rate of $0.5$ would mean that the number of correct digits increases by $1$ for each iteration (and this is what the bisection method provides). Here, convergence is slightly slower.

### Example 67. (cont'd)

(a) Newton's method applied to finding a root of $f(x) = x^3 - 2$ is equivalent to fixed-point iteration of which function $g(x)$?

(b) Does Newton's method converge locally to $\sqrt[3]{2}$? If so, determine the order and the rate.

This is a continuation of Example 60.

**Solution.**

(a) Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2}{3x^2} = \frac{2}{3}\left(x + \frac{1}{x^2}\right).$$

(b) By Theorem 57, Newton's method converges locally to $x^* = \sqrt[3]{2}$ if $|g'(x^*)| < 1$.

We compute that $g'(x) = \frac{2}{3} - \frac{4}{3x^3}$ so that $g'(x^*) = \frac{2}{3} - \frac{4}{3 \cdot 2} = 0$.

At this point, we know that Newton's method converges locally to $\sqrt[3]{2}$.

Moreover, $g''(x) = \frac{4}{x^4}$ so that $g''(x^*) = \frac{4}{2^{4/3}} = 2^{2/3} \approx 1.5874$.

Hence, the order of convergence is $2$ and the rate is $\frac{1}{2}|g''(x^*)| \approx 0.7937$.

**Comment.** Since the rate is less than $1$, the convergence is actually slightly better than a doubling of correct digits for each iteration.

**Important.** We will see shortly that it is typical for Newton's method to have convergence of order $2$.