Example 47. Apply Newton's method to $g(x) = x^3 - 2x + 2$ and initial value $x_0 = 0$.

Solution. Using $g'(x) = 3x^2 - 2$, we compute that $x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = 1$, $x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = 1 - \frac{1}{1} = 0$.

Since $x_2 = x_0$, the Newton method will now repeat and we are stuck in a 2-cycle.

In particular, the Newton method does not converge in this case.

Comment. $g(x)$ has one real root at $x \approx -1.7693$ (as well as two complex roots). Make a plot of $g(x)$!

Comment. It is possible to run into n-cycles for larger n as well when doing Newton iterations (for instance, try $f(x) = x^5 - x - 1$ and initial value $x_0 = 0$). When computing numerically, it is not particularly likely that we will run into a perfect cycle. However, such cycles can be **attractive**. Meaning that we get closer and closer to the cycle if we start with a nearby point. This is illustrated by the Python code experiment below.

Example 48. Python We apply the Newton method from our previous example to computing a root of $g(x) = x^3 - 2x + 2$. For that, we define g and its derivative as functions in Python:

```
>>> def g(x):
        return x**3 - 2*x + 2

>>> def gd(x):
        return 3*x**2 - 2
```

The following then confirms that we indeed have a 2-cycle starting with 0:

```
>>> [newton_method(g, gd, 0, n) for n in range(8)]

    [0, 1.0, 0.0, 1.0, 0.0, 1.0, 0.0, 1.0]
```

On the other hand, this is what happens if we start with a point close to 0:

```
>>> [newton_method(g, gd, 0.1, n) for n in range(8)]

    [0.1, 1.0142131979695432, 0.07965576631987636, 1.0090987403727651, 0.05222652653371296,
    1.0039651847274838, 0.02332943565497303, 1.0008043531824031]
```

Notice how we are being attracted by the 2-cycle.

Review: Taylor series

Recall from Calculus that, if $f(x)$ is **analytic** at $x = c$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \ldots$$

The series on the right-hand side is called the **Taylor series** of $f(x)$ at $x = c$.

Advanced comment. We only get the equality between $f(x)$ and its Taylor series for functions that are **analytic** at $x = c$ (by definition, these are functions than can be expanded as a power series; the above makes it explicit what the coefficients of that power series have to be). Fortunately, all elementary functions (the ones we can express as algebraic expressions with exponentials, logarithms and trig functions) are analytic at almost all points.

On the other hand, for instance, the Taylor series for the function $f(x) = e^{-1/x^2}$ at $x = 0$ is zero (because all derivatives of $f(x)$ are zero for $x = 0$) while $f(x)$ is not zero (however, note that $x = 0$ is clearly a problematic point of the formula for $f(x)$; that function is analytic at all other points). Since $f(x)$ is infinitely differentiable, this illustrates that being analytic is a stronger property than being infinitely differentiable. For other functions, it is possible that the Taylor series might not converge at all.

Comment. The Taylor series of $f(x)$ at $x = 0$ takes the form $f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots$. In practice, we can often shift things so that the Taylor series of interest are around $x = 0$.

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Example 49. Determine the Taylor series of e^x at 0.

Solution. If $f(x)=e^x$, then $f^{(n)}(x)=e^x$. In particular, we have $f^{(n)}(0)=1$ for all n.

Consequently, we have $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$

Comment. e^x is analytic everywhere and so it equals its Taylor series.

Example 50. Spell out the first few terms of the Taylor series of xe^{2x} at 0.

Solution. Since $e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$, we have $xe^{2x}=x\left(1+2x+\frac{1}{2}2^2x^2+\frac{1}{6}2^3x^3+\ldots\right)$ which simplifies to $xe^{2x}=x+2x^2+2x^3+\frac{4}{3}x^4+\ldots$.

Alternatively. We could set $f(x)=xe^{2x}$ and compute $f^{(n)}(0)$ for $n=0,1,2,\ldots$ to get our hands on the first few terms of the Taylor series.

Example 51. Determine the Taylor series of $\ln x$ at $x=1$.

Solution. If $f(x)=\ln(x)$, then $f'(x)=\frac{1}{x}$, $f''(x)=-\frac{1}{x^2}$, $f'''(x)=2\frac{1}{x^3}$, $f^{(4)}(x)=2\cdot 3\frac{1}{x^4}$, $\ldots$

For $n\geqslant 1$, we thus have $f^{(n)}(x)=(-1)^{n+1}\frac{(n-1)!}{x^n}$ and, in particular, $\frac{f^{(n)}(1)}{n!}=(-1)^{n+1}\frac{(n-1)!}{n!}=\frac{(-1)^{n+1}}{n}$.

Consequently, we have $\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$.

Comment. By replacing x with $1-x$, we obtain $\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x)=\ln\left(\frac{1}{1-x}\right)$.

If we take the derivative of both sides, we further find $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$. This is the famous **geometric series**.

The truncation $\sum_{n=0}^{M}\frac{f^{(n)}(c)}{n!}(x-c)^n$ is called the ***M*th Taylor polynomial** of $f(x)$ at $x=c$.

Comment. The Mth Taylor polynomial is a polynomial of degree at most M (note that the degree can be smaller if $f^{(M)}(c)=0$).

Important comment. The first Taylor polynomial of $f(x)$ at $x=c$ is the tangent line of $f(x)$ at $x=c$. In other words, it is the best linear approximation of $f(x)$ at $x=c$.

Likewise, the Mth Taylor polynomial is the best polynomial approximation at $x=c$ of degree up to M.

We have the following fundamental result for what happens when we truncate a Taylor series.

Theorem 52. (Taylor's theorem with error term) Suppose that $f(x)$ is $M+1$ times continuously differentiable on the interval between x and c. Then we have

$$f(x)\;=\;\underbrace{\sum_{n=0}^{M}\frac{f^{(n)}(c)}{n!}(x-c)^n}_{M\text{th Taylor polynomial}}\;+\;\underbrace{\frac{f^{(M+1)}(\xi)}{(M+1)!}(x-c)^{M+1}}_{\text{error term}}$$

for some ξ between x and c.

The special case $M=0$ of Taylor's theorem is equivalent to the mean value theorem:

Theorem 53. (mean value theorem) Suppose that $f(x)$ is differentiable on $[a,b]$. Then there exists $\xi\in(a,b)$ such that

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

Proof. Make a picture! □

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Note that Taylor's theorem provides us with a representation for the error when we approximate $f(x)$ with a Taylor polynomial. This is illustrated in the next example.

Example 54. Suppose we use the approximation $e^x \approx 1 + x + \frac{x^2}{2}$.

(a) Using Taylor's theorem, provide an upper bound for the error on the interval $[0, 1]$.

(b) Using Taylor's theorem, provide an upper bound for the error on the interval $[0, 0.1]$.

(c) Using Taylor's theorem, how many terms of the Taylor series do we need so that the error on $[0, 0.1]$ is less than 10^{-16}?

Solution. Note that $1 + x + \frac{x^2}{2}$ is the 2nd Taylor polynomial of $f(x) = e^x$ at $x = 0$.

(a) Taylor's theorem implies that

$$e^x - \left(1 + x + \frac{x^2}{2}\right) = \frac{f^{(3)}(\xi)}{3!}(x - 0)^3 = \frac{e^\xi}{3!}x^3$$

for some ξ between 0 and x.
Note that $|e^\xi| \leqslant e$ for all $\xi \in [0, 1]$.
On the other hand, $|x^3| \leqslant 1$ for all $x \in [0, 1]$.
We therefore conclude that the error on the interval $[0, 1]$ is bounded by

$$\left|e^x - \left(1 + x + \frac{x^2}{2}\right)\right| = \left|\frac{e^\xi}{3!}x^3\right| \leqslant \frac{e}{3!} \approx 0.453.$$

Comment. In this simple case, we can determine the maximal error exactly (without using Taylor's theorem). Since the function $e^x - \left(1 + x + \frac{x^2}{2}\right)$ is increasing on the interval $[0, 1]$, starting with the value 0, the maximal error must occur at $x = 1$ and is $e - \frac{5}{2} \approx 0.218$. We thus find that our earlier error bound was a bit conservative but not a bad upper bound.

(b) As above, we conclude that the error on the interval $[0, 0.1]$ is bounded by

$$\left|e^x - \left(1 + x + \frac{x^2}{2}\right)\right| = \left|\frac{e^\xi}{3!}x^3\right| \leqslant \frac{e^{0.1}}{3!}0.1^3 \approx 0.000184 = 1.84 \cdot 10^{-4}.$$

Comment. For comparison, as above, the maximal actual error is $1.71 \cdot 10^{-4}$.

(c) By Taylor's theorem,

$$|e^x - p_M(x)| = \left|\frac{e^\xi}{(M+1)!}x^{M+1}\right| \leqslant \frac{e^{0.1}}{(M+1)!}0.1^{M+1}.$$

We wish to choose M so that the right-hand side is less than 10^{-16}. Since the right-hand side decreases very rapidly, we simply increase M until that happens:

$$\frac{e^{0.1}}{3!}0.1^3 \approx 1.8 \cdot 10^{-4}, \quad \ldots, \quad \frac{e^{0.1}}{9!}0.1^9 \approx 3.0 \cdot 10^{-15}, \quad \frac{e^{0.1}}{10!}0.1^{10} \approx 3.0 \cdot 10^{-17}.$$

We conclude that the 9th Taylor polynomial will approximate e^x in such a way that the error on $[0, 0.1]$ is less than 10^{-16}.

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