Final: practice

MATH 436/565 — Numerical Analysis
final exam: Wednesday, Dec 11, 2023

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2. Consider the initial value problem $y' = xy^2 + 1,\ y(1) = 0$.

(a) Approximate the solution $y(x)$ for $x \in [1, 3]$ using Euler's method with 3 steps. In particular, what is the approximation for $y(3)$?

(b) What is the order of the local truncation error? The global error?

(c) Spell out the Taylor method of order 3 for numerically solving this initial value problem.

Solution.

(a) The step size is $h = \frac{3-1}{3} = \frac{2}{3}$. We apply Euler's method with $f(x, y) = xy^2 + 1$:

$$\begin{array}{ll} x_0 = 1 & y_0 = 0 \\ x_1 = \dfrac{5}{3} & y_1 = y_0 + f(x_0, y_0)h = \dfrac{2}{3} \\ x_2 = \dfrac{7}{3} & y_2 = y_1 + f(x_1, y_1)h = \dfrac{2}{3} + \left(\dfrac{5}{3}\left(\dfrac{2}{3}\right)^2 + 1\right)\dfrac{2}{3} = \dfrac{148}{81} \\ x_3 = 3 & y_3 = y_2 + f(x_2, y_2)h = \dfrac{148}{81} + \left(\dfrac{7}{3}\left(\dfrac{148}{81}\right)^2 + 1\right)\dfrac{2}{3} = \dfrac{453914}{59049} \end{array}$$

In particular, the approximation for $y(3)$ is $y_3 = \frac{453914}{59049} \approx 7.68707$.

(b) The local truncation error is of order 2, and the global error is of order 1.

In other words, the local truncation error is $O(h^2)$ and the global error is $O(h)$, where h is the step size.

(c) The Taylor method of order 3 is based on the Taylor expansion

$$y(x + h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2 + \frac{1}{6}y'''(x)h^3 + O(h^4),$$

where we have a local truncation error of $O(h^4)$ so that the global error will be $O(h^3)$.

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From the DE we know that $y'(x) = xy^2 + 1$, which is $f(x, y)$. We differentiate this to obtain

$$y''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(xy^2 + 1) = y^2 + 2xyy' = y^2 + 2xy \cdot (xy^2 + 1) = y^2 + 2x^2y^3 + 2xy$$

which is $f'(x, y)$. We differentiate once more to find

$$\begin{aligned} y'''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(y^2 + 2x^2y^3 + 2xy) = 2yy' + 4xy^3 + 6x^2y^2y' + 2y + 2xy' \\ &= (2y + 6x^2y^2 + 2x)(xy^2 + 1) + 4xy^3 + 2y \\ &= 2(x + 2y + 4x^2y^2 + 3xy^3 + 3x^3y^4) \end{aligned}$$

which is $f''(x, y)$ (in the second step we replaced y' by $xy^2 + 1$).

Hence, the Taylor method of order 3 takes the form:

$$\begin{aligned} y_{n+1} &= y_n + f(x_n, y_n)h + \tfrac{1}{2}f'(x_n, y_n)h^2 + \tfrac{1}{6}f''(x_n, y_n)h^3 \\ &= y_n + (x_ny_n^2 + 1)h + \tfrac{1}{2}(y_n^2 + 2x_n^2y_n^3 + 2x_ny_n)h^2 + \tfrac{1}{3}(x_n + 2y_n + 4x_n^2y_n^2 + 3x_ny_n^3 + 3x_n^3y_n^4)h^3 \end{aligned}$$