

Midterm #2 – Practice

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Determine the minimal polynomial $P(x)$ interpolating $(-2, 1), (0, 1), (1, 1), (3, 2)$.

- Write down the polynomial in Lagrange form.
- Write down the polynomial in Newton form.
- Suppose the above points lie on the graph of a smooth function $f(x)$. Write down an “explicit” formula for $f(x) - P(x)$, the error when using the interpolating polynomial to approximate $f(x)$.

Problem 2. Suppose we approximate $f(x) = \cos(\frac{x}{2})$ by the polynomial $P(x)$ interpolating it at $x = 1, 2, 3$.

- Without computing $P(x)$, give an upper bound for the error when $x = 0$ and when $x = \frac{\pi}{2}$.
- For which x in $[0, \pi]$ is our bound for the error maximal? What is the bound in that case?

Problem 3. Suppose we approximate a function $f(x)$ by the polynomial $P(x)$ interpolating it at $x = -1, -\frac{2}{3}, \frac{2}{3}, 1$. Suppose that we know that $|f^{(n)}(x)| \leq n$ for all $x \in [-1, 1]$.

- Give an upper bound for the error when $x = -\frac{1}{6}$ and when $x = 0$.
- Give an upper bound for the error for all $x \in [-1, 1]$.
- Suppose we replace the nodes $-1, -\frac{2}{3}, \frac{2}{3}, 1$ with four other values. For which choice of these four interpolation nodes, is this upper bound for the error minimal?
- For this optimal choice, what is the upper bound for the error for all $x \in [-1, 1]$?

Problem 4. Suppose that $f(x)$ is a smooth function such that $|f^{(n)}(x)| \leq n!$ for all $x \in [-1, 1]$ and all $n \geq 0$. Suppose we approximate $f(x)$ on the interval $[-1, 1]$ by a polynomial interpolation $P(x)$. How many Chebyshev nodes do we need to use in order to guarantee that the maximal error is at most 10^{-6} ?

Problem 5. Determine the natural cubic spline through $(-3, 1), (0, 3), (2, 1)$.

Problem 6. Recall that a cubic spline $S(x)$ through $(x_0, y_0), \dots, (x_n, y_n)$ with $x_0 < x_1 < \dots < x_n$ is piecewise defined by n cubic polynomials $S_1(x), \dots, S_n(x)$ such that $S(x) = S_i(x)$ for $x \in [x_{i-1}, x_i]$. Name three common boundary conditions of cubic splines and state their mathematical definition.

Problem 7. Obtain approximations for $f'(x)$ and $f''(x)$ using the values $f(x - 2h)$, $f(x)$, $f(x + 3h)$ as follows: determine the polynomial interpolation corresponding to these values and then use its derivatives to approximate those of f . In each case, determine the order of the approximation and the leading term of the error.

Problem 8. Suppose that $A(\frac{1}{4}) = a$ and $A(\frac{1}{10}) = b$ are approximations of order 4 of some quantity A^* . What is the approximation we obtain from using Richardson extrapolation?

Problem 9. We have shown that $A(h) = \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)]$ is an approximation of $f''(x)$ of order 2.

- (a) Determine the leading term of the error.
- (b) Apply Richardson extrapolation to $A(h)$ and $A(3h)$ to obtain an approximation of $f''(x)$ of higher order.
- (c) Explain in a sentence why the resulting approximation is of order 4 (rather than 3).