Midterm #1

MATH 436 — Numerical Analysis
Thursday, Sept 29

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 36 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points) Consider $f(x) = (x + r)(x^2 - 1)$ where r is some constant. Suppose we want to use Newton's method to calculate the root $x^* = 1$.

(a) For which values of r is Newton's method guaranteed to converge (at least) quadratically to $x^* = 1$?

(b) For which values of r does Newton's method converge to $x^* = 1$ faster than quadratically?

Solution. Recall that we showed that, if $f(x^*) = 0$ and $f'(x^*) \neq 0$, then Newton's method (locally) converges to x^* quadratically with rate $\frac{1}{2}|f''(x^*) / f'(x^*)|$.

(a) Newton's method is guaranteed to converge to 1 provided that $f'(1) \neq 0$.

Since $f(1) = 0$, we have $f'(1) = 0$ if and only if 1 is a repeated root which happens if and only if $r = -1$.

Alternatively, we could compute $f'(x) = (x^2 - 1) + 2x(x + r)$ so that $f'(1) = 2(r + 1)$. Thus $f'(1) = 0$ if and only if $r = -1$.

In either case, we conclude that Newton's method converges (at least) quadratically to $x^* = 1$ if $r \neq -1$.

(b) Newton's method converges to 1 faster than quadratic if $f'(1) \neq 0$ (i.e. $r \neq -1$) and $f''(1) = 0$.

Continuing the computation from the first part, we calculate $f''(x) = 6x + 2r$.

Thus $f''(1) = 6 + 2r = 0$ if and only if $r = -3$.

Hence, Newton's method converges to 1 faster than quadratic if $r = -3$.

Problem 2. (3 points)

(a) Give one advantage of the secant method over the regula falsi method.

(b) Give one advantage of the regula falsi method over the secant method.

Solution.

(a) An advantage of the secant method is that, if it converges, it typically converges faster than linear. (Also, it does not require an initial interval that is guaranteed to contain a root.)

(b) An advantage of the regula falsi method is that it is guaranteed to converge.

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Problem 3. (6 points) Determine all fixed-points of $f(x) = \dfrac{x}{x+2}$. For each fixed-point x^* determine whether fixed-point iteration of $f(x)$ converges locally to x^*. If so, determine the exact order of convergence as well as the rate.

Solution. Such values of C are necessarily fixed points of $f(x)$. To find these, we solve $\frac{x}{x+2} = x$.

The resulting fixed points are $x^* = 0$ and $x^* = -1$.

$f'(x) = \frac{(x+2) - x}{(x+2)^2} = \frac{2}{(x+2)^2}$

- $f'(0) = \frac{1}{2}$

  Since $|f'(0)| < 1$, fixed-point iteration converges locally to 0. The convergence is linear with rate $|f'(0)| = \frac{1}{2}$.

- $f'(-1) = 2$

  Since $|f'(-1)| > 1$, fixed-point iteration does not converge locally to -1.

Problem 4. (3 points) Express $19/6$ in base 2. If necessary, indicate which digits repeat.

Solution. Note that $19/6 = 3 + 1/6$ so that $19/6 = (11.\cdots)_2$ with $1/6$ to be accounted for.

- $2 \cdot 1/6 = \boxed{0} + 1/3$

- $2 \cdot 1/3 = \boxed{0} + 2/3$

- $2 \cdot 2/3 = \boxed{1} + 1/3$ and now things repeat…

Hence, $19/6 = (11.001\cdots)_2$ and the final two digits 01 repeat: $19/6 = (11.0010101\cdots)_2$

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Problem 5. (8 points) We wish to compute the root $\sqrt{3}$ of $f(x) = x^2 - 3$ using the bisection method.

(a) Starting with the interval $[1, 2]$, apply two iterations of bisection. What is the resulting approximation of $\sqrt{3}$?

(b) After how many iterations can we guarantee that the absolute error is less than 0.001?

(c) Describe in a few words how the regula falsi method proceeds different from the bisection method.

(d) Newton's method applied to $x^2 - 3$ is equivalent to fixed-point iteration of which function $g(x)$?

Solution.

(a) Note that $f(1) = -2 < 0$ while $f(2) = 1 > 0$. Hence, $f(x)$ must indeed have a root in the interval $[1, 2]$.

- The midpoint of $[1, 2]$ is $\frac{1+2}{2} = \frac{3}{2}$. Since $f\left(\frac{3}{2}\right) = \frac{9}{4} - 3 = -\frac{3}{4} < 0$, a root of $f(x)$ must be in $\left[\frac{3}{2}, 2\right]$.

- The midpoint of $\left[\frac{3}{2}, 2\right]$ is $\frac{3/2+2}{2} = \frac{7}{4}$. Since $f\left(\frac{7}{4}\right) = \frac{49}{16} - 3 = \frac{1}{16} > 0$, a root of $f(x)$ must be in $\left[\frac{3}{2}, \frac{7}{4}\right]$.

The best approximation of $\sqrt{3} \approx 1.732$ at this point is the midpoint of the final interval: $\frac{13}{8} \approx 1.625$

(b) The width of the interval after n steps will be exactly $\ell = \frac{2-1}{2^n} = \frac{1}{2^n}$. Since $\sqrt{3}$ is contained in this interval, the absolute error of approximating it with the midpoint is at most $\ell/2 = \frac{1}{2^{n+1}}$. We need to select n so that $\frac{1}{2^{n+1}} < 10^{-3}$. Knowing that $2^{10} = 1024$ (and $\frac{1}{1024} < \frac{1}{1000}$), we conclude that we need 9 iterations.

(c) The regula falsi method proceeds like the bisection method. However, instead of using the midpoint $\frac{a+b}{2}$ of the interval $[a, b]$, it uses the root of the secant line of $f(x)$ through $(a, f(a))$ and $(b, f(b))$.

(d) Newton's method applied to $f(x) = x^2 - 3$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2-3}{2x} = \frac{x}{2} + \frac{3}{2x}.$$

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Problem 6. (2 points)

(a) Suppose we use the regula falsi method to compute the root of a function $f(x)$. Several iterations result in the intervals $[1, 2], \left[\frac{7}{6}, 2\right], \left[\frac{44}{37}, 2\right], \left[\frac{273}{229}, 2\right]$.

Based on these, our approximation of the root is ☐.

(b) Express -18 in binary using the two's complement representation with 6 bits.

☐

Solution.

(a) Our approximation of the root is the most recently updated endpoint: $\frac{273}{229}$

(By the way, the intervals were generated for $f(x) = x^2 + 3x - 5$.)

(b) Since $18 = (010010)_2$, -18 is represented by 101110 (invert all bits, then add 1).

Alternatively, note that $-18 = -2^5 + 14$ and $14 = (1110)_2$ to arrive at the same representation.

Problem 7. (5 points) Suppose we wish to approximate the function $f(x) = 2x\ln(x)$.

(a) What is the 2nd Taylor polynomial $p_2(x)$ of $f(x)$ at $x = 1$?

(b) Provide an upper bound for the error of approximating $f(x)$ by $p_2(x)$ on the interval $[1, 2]$.

Solution.

(a) $f'(x) = 2\ln(x) + 2$

$f''(x) = \frac{2}{x}$

Hence, the 2nd Taylor polynomial of $f(x)$ at $x = 1$ is

$$p_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2(x-1) + (x-1)^2.$$

Comment. There is typically no reason to expand this out since this approximation is intended to be used for x of the form $1 + \delta$, where δ is small (i.e. for x close to 1).

(b) Taylor's theorem implies that

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-1)^3$$

for some ξ between 1 and x.

We compute that $f'''(x) = -\frac{2}{x^2}$. This function is increasing on $(0, \infty)$ and so, in particular, on $[1, 2]$. Therefore, the maximum absolute value on $[1, 2]$ is taken at $x = 1$ or $x = 2$. Since $|f'''(1)| = 2$ and $|f'''(2)| = \frac{1}{2}$, we conclude that $|f'''(\xi)| \leqslant 2$.

On the other hand, $|(x-1)^3| \leqslant 1^3 = 1$ for all $x \in [1, 2]$.

We therefore conclude that the error on $[1, 2]$ is bounded by

$$|f(x) - p_2(x)| = \left|\frac{f^{(3)}(\xi)}{3!}(x-1)^3\right| \leqslant \frac{2}{3!} \cdot 1^3 = \frac{1}{3}.$$

Comment. In this simple case, we can determine the maximal error exactly (without using Taylor's theorem). Since the function $f(x) - p_2(x)$ is decreasing on the interval $[1, 2]$, starting with the value 0, the maximal error must occur at $x = 2$ and is $|4\ln(2) - 3| \approx 0.227$.

Problem 8. (3 points) Represent -2.5 as a single precision floating-point number according to IEEE 754.

Solution. $-2.5 = -1.25 \cdot 2^1 = \boxed{-}1.\underbrace{\boxed{01}}_{\text{binary}} \cdot 2^1$

The exponent 1 gets stored as $1 + 127 = \boxed{1000{,}0000}$.

Overall, -2.5 is stored as $\boxed{1}\boxed{1000{,}0000}\boxed{0100{,}0000{,}0000{,}0000{,}0000{,}000}$.

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(extra scratch paper)