Example 81. Determine the minimal polynomial interpolating $(0,1),(1,2),(2,5)$.

Solution. (Lagrange, review) The interpolating polynomial in Lagrange form is:

$$\begin{aligned} p(x) &= 1\frac{(x-1)(x-2)}{(0-1)(0-2)} + 2\frac{(x-0)(x-2)}{(1-0)(1-2)} + 5\frac{(x-0)(x-1)}{(2-0)(2-1)} \\ &= \frac{1}{2}(x-1)(x-2) - 2x(x-2) + \frac{5}{2}x(x-1) \\ &= x^2+1 \end{aligned}$$

Solution. (Newton, divided differences)

$$\begin{array}{llll} 0: & 1 & & \\ & & \frac{2-1}{1-0}=1 & \\ 1: & 2 & & \frac{3-1}{2-0}=1 \\ & & \frac{5-2}{2-1}=3 & \\ 2: & 5 & & \end{array}$$

Accordingly, reading the coefficients from the top edge of the triangle:

$$p(x) = 1 + 1(x-0) + 1(x-0)(x-1) = x^2+1$$

A mean value theorem for divided differences

Review. The **mean value theorem** (see Theorem 49; the special case $M=0$ of Taylor's theorem) states that, if $f(x)$ is differentiable, then

$$f[a,b] = \frac{f(b)-f(a)}{b-a} = f'(\xi)$$

for some ξ between a and b.

Recall that the Newton form of the polynomial interpolating $f(x)$ at $x = x_0, x_1, \ldots$ is

$$f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2) + \ldots$$

Note that this is somewhat similar to the Taylor expansion of $f(x)$ at $x = x_0$, which is

$$f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2!}f''(x_0)(x-x_0)^2 + \tfrac{1}{3!}f'''(x_0)(x-x_0)^3 + \ldots$$

Indeed, if all the x_j are equal to x_0 (this is technically not allowed when interpolating, but you can still think of choosing them all close to x_0), then the Newton form would turn into a Taylor polynomial.
In that case, $f[x_0,x_1,\ldots,x_n]$ would become $\frac{1}{n!}f^{(n)}(x_0)$.

With that (as well as the mean value theorem and Taylor's theorem (see Theorem 48)) in mind, the next result does not come as a surprise.

Theorem 82. (mean value theorem for divided differences) If $f(x)$ is differentiable, then

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some ξ between the smallest and the largest of the x_i.

Proof. Without loss of generality, we may assume that $x_0 < x_1 < \cdots < x_n$ (because divided differences do not depend on the ordering of the points x_i).

Let $P(x)$ be the interpolation polynomial for f at $x_0, x_1, \ldots, x_n$. Then $d(x) = f(x) - P(x)$ has $n+1$ zeros, namely $x_0, x_1, \ldots, x_n$. The mean value theorem implies that between any two zeros of a function, there must be a zero of its derivative (this is often referred to as Rolle's theorem). It therefore follows that $d'(x)$ has n zeros (between x_0 and x_n). Applying the same argument to $d'(x)$, we then find that $d''(x)$ has $n-1$ zeros. Continuing like this, $d^{(n)}(x)$ must have a zero ξ between x_0 and x_n. As such,

$$0 = d^{(n)}(\xi) = f^{(n)}(\xi) - P^{(n)}(\xi).$$

Recall that $P(x)$ is a polynomial of degree n or less, and that its Newton form is

$$P(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

where $c_j = f[x_0, x_1, \ldots, x_j]$. Note that $P^{(n)}(x) = n!c_n = n!\,f[x_0, x_1, \ldots, x_n]$. We therefore conclude that

$$0 = d^{(n)}(\xi) = f^{(n)}(\xi) - P^{(n)}(\xi) = f^{(n)}(\xi) - n!\,f[x_0, x_1, \ldots, x_n],$$

which proves the claim. □

Comment. Note that this provides us with a way to numerically approximate an nth derivative $f^{(n)}(x)$. Namely, choose $n+1$ points $x_0, x_1, \ldots, x_n$ near x. Then $f^{(n)}(x) \approx \underbrace{n!\,f[x_0, x_1, \ldots, x_n]}_{= f^{(n)}(\xi)}$.

Bounding the interpolation error

Theorem 83. (interpolation error) Suppose that $f(x)$ is $n+1$ times continuously differentiable. Let $P_n(x)$ be the interpolating polynomial for $f(x)$ at $x_0, x_1, \ldots, x_n$. Then

$$f(x) - P_n(x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)}_{\text{interpolation error}}$$

for some ξ between the smallest and the largest of the x_i together with x.

Proof. Let $P_{n+1}(x)$ be the interpolating polynomial for $f(x)$ at $x_0, x_1, \ldots, x_n, x_{n+1}$. We know that

$$\begin{aligned} P_{n+1}(x) &= P_n(x) + f[x_0, x_1, \ldots, x_{n+1}](x - x_0)(x - x_1)\cdots(x - x_n) \\ &= P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \end{aligned}$$

for some ξ between the smallest and the largest of the x together with x.

Given any fixed value t, choose $x_{n+1} = t$ in this formula (so that $P_{n+1}(t) = f(t)$) to conclude that

$$f(t) = P_{n+1}(t) = P_n(t) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - x_0)(t - x_1)\cdots(t - x_n),$$

which is the claimed expression for the error term (with x replaced by t). □

Example 84. Suppose we approximate $f(x) = \sin(x)$ by the polynomial $P(x)$ interpolating it at $x = 0, \frac{\pi}{2}, \pi$. Without computing $P(x)$, give an upper bound for the error when $x = \frac{\pi}{4}$.

[Compare with Example 78 where we computed and plotted $P(x)$.]

Solution. By Theorem 83, the error is

$$\sin(x) - P(x) = \frac{f^{(3)}(\xi)}{3!}(x-0)\left(x - \frac{\pi}{2}\right)(x-\pi),$$

where ξ is between 0 and π (provided that x is in $[0, \pi]$). Note that $f^{(3)}(x) = -\cos(x)$ so that $|f^{(3)}(\xi)| \leqslant 1$. Hence, the error is bounded by

$$|\sin(x) - P(x)| \leqslant \frac{1}{6}\left|x\left(x - \frac{\pi}{2}\right)(x-\pi)\right|.$$

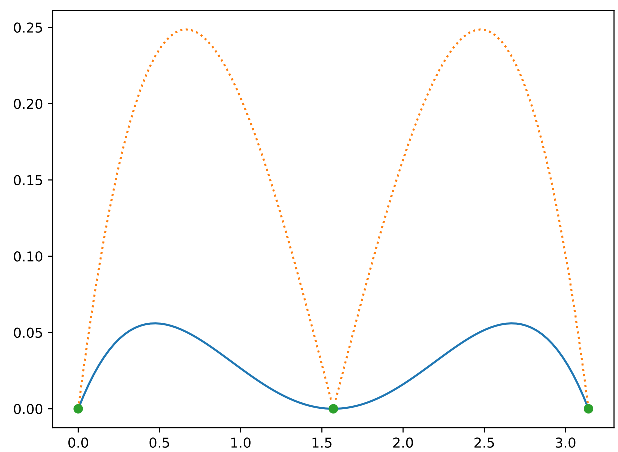
In particular, in the case $x = \frac{\pi}{4}$,

$$\left|\sin\left(\frac{\pi}{4}\right) - P\left(\frac{\pi}{4}\right)\right| \leqslant \frac{1}{6}\left|\frac{\pi}{4}\left(-\frac{\pi}{4}\right)\left(-\frac{3\pi}{4}\right)\right| = \frac{\pi^3}{128} \approx 0.242.$$

For comparison. In this particularly simple case, we can easily calculate the exact error.

Namely, since $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $P\left(\frac{\pi}{4}\right) = \frac{3}{4}$ (see Example 78), the actual error is $\left|\sin\left(\frac{\pi}{4}\right) - P\left(\frac{\pi}{4}\right)\right| \approx 0.0428$.

Below is a plot of the actual error (in blue) together with our bound (dotted).



Homework. Following what we did in Example 78, try to reproduce this plot.

For which x in $[0, \pi]$ is our bound for the error maximal? What is the bound in that case?

Solution. Recall that our bound for the error is $\frac{1}{6}\left|x\left(x - \frac{\pi}{2}\right)(x-\pi)\right|$.

$x\left(x - \frac{\pi}{2}\right)(x-\pi)$ is maximal on $[0, \pi]$ for $x = \left(1 \pm \frac{1}{\sqrt{3}}\right)\frac{\pi}{2} \approx 0.664, 2.478$. (Fill in the details!)

The corresponding error bound is $\frac{1}{72\sqrt{3}}\pi^3 \approx 0.249$.

Comment. Note that this shows that our earlier error bound for $x = \frac{\pi}{4} \approx 0.785$ was close to maximal. That is not too much of a surprise since $\frac{\pi}{4}$ sits right between 0 and $\frac{\pi}{2}$ for which the error is 0 by construction.

For comparison. The actual maximal error occurs when $\cos(x) - \frac{4}{\pi} + \frac{8}{\pi^2}x = 0$. (Why?!)

The approximate solutions are $x \approx 0.472, 2.670$ with corresponding (actual) error of 0.0560.

Make sure that you can identify both the x values and the error in the above plot.