Order of convergence of fixed-point iteration

Theorem 61. Suppose that x^* is a fixed point of a continuously differentiable function f. Suppose that $|f'(x^*)| < 1$ so that, by Theorem 53, fixed-point iteration of $f(x)$ converges to x^* locally.

Then the convergence is of order M with rate $\frac{1}{M!}|f^{(M)}(x^*)|$ where $M \geqslant 1$ is the smallest integer so that $f^{(M)}(x^*) \neq 0$.

In particular.

- If $f'(x^*) \neq 0$, then the convergence is linear with rate $|f'(x^*)|$.
- If $f'(x^*) = 0$ and $f''(x^*) \neq 0$, then the convergence is quadratic with rate $\frac{1}{2}|f''(x^*)|$.

Proof. By Taylor's theorem, if $f'(x^*) = f''(x^*) = \cdots = f^{(M-1)}(x^*) = 0$ for some $M \geqslant 1$, then

$$f(x) = f(x^*) + \frac{1}{M!} f^{(M)}(\xi)(x - x^*)^M$$

for some ξ between x and x^*. It follows that

$$\begin{aligned} x_{n+1} - x^* &= f(x_n) - f(x^*) \\ &= \frac{1}{M!} f^{(M)}(\xi_n)(x_n - x^*)^M \end{aligned}$$

for some ξ_n between x_n and x^*.
Thus

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^M} = \frac{1}{M!} f^{(M)}(\xi_n) \quad \underset{n \to \infty}{\longrightarrow} \quad \frac{1}{M!} f^{(M)}(x^*),$$

where the limit follows from the continuity of $f^{(M)}(x)$ (and convergence of $x_n \to x^*$). □

Applying fixed-point iteration directly

Note that any equation $f(x) = 0$ can be rewritten in many ways as a fixed-point equation $g(x) = x$.

For instance. We can always rewrite $f(x) = 0$ as $f(x) + x = x$ (i.e. choose $g(x) = f(x) + x$).

We can then attempt to find a root x^* of $f(x)$ by fixed-point iteration on $g(x)$.

In other words, we start with a value x_0 (an initial approximation) and then compute $x_1, x_2, \ldots$ via $x_{n+1} = g(x_n)$.

Theorem 61 tells us whether that such a fixed-point iteration on $g(x)$ will locally converge to x^*. Moreover, it tells us the order of convergence.

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Example 62. Suppose we are interested in computing the roots of $x^2 - x - 1 = 0$.

The roots are the golden ratio $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ and $\psi = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618$.

There are many ways to rewrite this equation as a fixed-point equation $g(x) = x$. The following are three possibilities:

(a) Rewrite as $x = x^2 - 1$, so that $g(x) = x^2 - 1$.

(b) Rewrite first as $x^2 = x + 1$ and then as $x = 1 + \frac{1}{x}$, so that $g(x) = 1 + \frac{1}{x}$.

(c) Rewrite first as $\underbrace{x^2 - x}_{= x(x-1)} = 1$ and then as $x = \frac{1}{x-1}$, so that $g(x) = \frac{1}{x-1}$.

In each of these three cases and for each root, decide whether fixed-point iteration converges. If it does, determine the order and rate of convergence.

Solution.

(a) In this case, we have $g(x) = x^2 - 1$ and $g'(x) = 2x$.
Since $|g'(\phi)| \approx 3.236 > 1$ as well as $|g'(\psi)| \approx 1.236 > 1$, fixed-point iteration does not converge locally to either root.

(b) In this case, we have $g(x) = 1 + \frac{1}{x}$ and $g'(x) = -\frac{1}{x^2}$.
Since $|g'(\phi)| = \frac{1}{\phi + 1} \approx 0.382 < 1$ and $|g'(\psi)| = \phi + 1 \approx 2.618 > 1$, fixed-point iteration converges locally to ϕ but does not converge locally to ψ. Moreover, the convergence to ϕ is linear with rate 0.382.

(c) In this case, we have $g(x) = \frac{1}{x-1}$ and $g'(x) = -\frac{1}{(x-1)^2}$.
Since $|g'(\phi)| = \phi + 1 \approx 2.618 > 1$ and $|g'(\psi)| = \frac{1}{\phi + 1} \approx 0.382 < 1$, fixed-point iteration converges locally to ψ but does not converge locally to ϕ. Moreover, the convergence to ψ is linear with rate 0.382.

Order of convergence of Newton's method

Recall that computing a root x^* of $f(x)$ using Newton's method is equivalent to fixed-point iteration of $g(x) = x - \frac{f(x)}{f'(x)}$.

Comment. In each case, we start with x_0 and iteratively compute $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Theorem 63. If $f(x^*) = 0$ and $f'(x^*) \neq 0$, then Newton's method (locally) converges to x^* quadratically with rate $\frac{1}{2}|f''(x^*) / f'(x^*)|$.

Comment. If, in addition, $f''(x^*) = 0$, then Newton's method even converges with order at least 3.

Comment. On the other hand, if $f(x^*) = 0$ and $f'(x^*) = 0$, then x^* is a repeated root of $f(x)$. In that problematic case, Newton's method either does not converge at all or it converges linearly.

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Proof. We apply Theorem 61 to analyze the fixed-point iteration of $g(x) = x - \frac{f(x)}{f'(x)}$.
Using the quotient rule we compute that

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

If $f(x^*) = 0$ and $f'(x^*) \neq 0$, then we have $g'(x^*) = 0$. By Theorem 61 this implies that fixed-point iteration converges at least quadratically.
To determine the rate of convergence, we further compute (again using the quotient and product rule) that

$$g''(x) = \frac{(f'(x)f''(x) + f(x)f'''(x))f'(x)^2 - 2f(x)f''(x)f'(x)f''(x)}{f'(x)^4}.$$

From this (unsimplified) expression and $f(x^*) = 0$ we conclude that $g''(x^*) = \frac{f''(x^*)}{f'(x^*)}$.

By Theorem 61 this implies that the convergence is quadratic with rate $\frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right|$.

Moreover, if $f''(x^*) = 0$ then $g''(x^*) = 0$ so that the convergence is even cubic (or higher). □

Example 64. $f(x) = e^{-x} - x$ has the unique root $x^* \approx 0.567$. Determine whether Newton's method converges locally to x^*. If it does, what is the order and rate of convergence?

Solution. We compute that $f'(x) = -e^{-x} - 1$ and $f''(x) = e^{-x}$.
Since $x^* = e^{-x^*}$, we have $f'(x^*) = -x^* - 1 \neq 0$.
Hence, by Theorem 63, Newton's method converges to x^* quadratically.
Moreover, the rate is $\frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right| = \frac{1}{2}\left|\frac{e^{-x^*}}{-e^{-x^*} - 1}\right| = \frac{1}{2}\left|\frac{x^*}{-x^* - 1}\right| \approx 0.181$.

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