Example 54. From a plot of $\cos(x)$, we can see that it has a unique fixed point in the interval $[0, 1]$.

Solution. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. Since $|\sin(x)| < 1$ for all $x \in [0, 1]$, we conclude that $|f'(x^*)| < 1$. By Theorem 53, fixed-point iteration will therefore converge to x^* locally.

Example 55. Python Let us implement the fixed-point iteration of $\cos(x)$ from the previous example in Python.

```
>>> from math import cos

>>> def cos_iterate(x, n):
        for i in range(n):
            x = cos(x)
        return x

>>> [cos_iterate(1, n) for n in range(20)]
```

```
[1, 0.5403023058681398, 0.8575532158463934, 0.6542897904977791, 0.7934803587425656,
0.7013687736227565, 0.7639596829006542, 0.7221024250267077, 0.7504177617637605,
0.7314040424225098, 0.7442373549005569, 0.7356047404363474, 0.7414250866101092,
0.7375068905132428, 0.7401473355678757, 0.7383692041223232, 0.7395672022122561,
0.7387603198742113, 0.7393038923969059, 0.7389377567153445]
```

Comment. Instead of using a loop, we could also implement the above fixed-point iteration **recursively** in the following way (the recursive part is that the function is calling itself).

```
>>> def cos_iterate_recursively(x, n):
        if n > 0:
            return cos_iterate_recursively(cos(x), n-1)
        return x

>>> [cos_iterate_recursively(1, n) for n in range(20)]
```

```
[1, 0.5403023058681398, 0.8575532158463934, 0.6542897904977791, 0.7934803587425656,
0.7013687736227565, 0.7639596829006542, 0.7221024250267077, 0.7504177617637605,
0.7314040424225098, 0.7442373549005569, 0.7356047404363474, 0.7414250866101092,
0.7375068905132428, 0.7401473355678757, 0.7383692041223232, 0.7395672022122561,
0.7387603198742113, 0.7393038923969059, 0.7389377567153445]
```

Sometimes recursion results in cleaner code. However the use of loops is usually more efficient.

Newton's method as a fixed-point iteration

Recall that Newton's method for finding a root of $f(x)$ proceeds from an initial approximation x_0 and iteratively computes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Note that this is equivalent to fixed-point iteration of the function $g(x) = x - \frac{f(x)}{f'(x)}$.

Comment. Note that x^* is a fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$ if and only if $\frac{f(x^*)}{f'(x^*)} = 0$.

We have already proven a criterion for convergence of fixed-point iterations (Theorem 53). Our next goal is to develop the tools to analyze the speed of that convergence.

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Example 56.

(a) Newton's method applied to finding a root of $f(x) = x^3 - 2$ is equivalent to fixed-point iteration of which function $g(x)$?

(b) Determine whether Newton's method converges locally to $\sqrt[3]{2}$.

Solution.

(a) Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2}{3x^2} = \frac{2}{3}\left(x + \frac{1}{x^2}\right).$$

(b) By Theorem 53, Newton's method converges locally to $x^* = \sqrt[3]{2}$ if $|g'(x^*)| < 1$.

We compute that $g'(x) = \frac{2}{3} - \frac{4}{3x^3}$ so that $g'(x^*) = \frac{2}{3} - \frac{4}{3 \cdot 2} = 0$.

Hence Newton's method converges locally to $\sqrt[3]{2}$.

Important comment. Notice that $g'(x^*) = 0$ is, in a way, the strongest sense in which $|g'(x^*)| < 1$. We will see shortly that $g'(x^*) = 0$ implies especially fast convergence of the type we observed in Example 41.

Example 57.

(a) What are the fixed points of $g(x) = \frac{x}{2} + \frac{1}{x}$?

(b) Does fixed-point iteration of $g(x)$ converge?

(c) Find a function $f(x)$ such that the fixed-point iteration of $g(x)$ is equivalent to Newton's method applied to $f(x)$.

(d) Inspired by the previous parts, suggest a fixed-point iteration to compute square roots.

Solution.

(a) Solving $\frac{x}{2} + \frac{1}{x} = x$, we find $x^2 = 2$ and thus $x = \pm\sqrt{2}$.

Comment. Note that $g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$. Suppose that $x < \sqrt{2}$. Then $2/x > \sqrt{2}$.

When iterating $g(x)$, we are averaging the underestimate and the overestimate, and it is reasonable to expect that the result is a better approximation.

(b) Since $g'(x) = \frac{1}{2} - \frac{1}{x^2}$, we have $g'(\pm\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0$. Hence, both fixed points are attracting fixed points.

By Theorem 53, fixed-point iteration of $g(x)$ converges locally to both fixed points.

(c) We are looking for a function $f(x)$ such that $x - \frac{f(x)}{f'(x)} = g(x)$. Equivalently, $\frac{f'(x)}{f(x)} = \frac{1}{x - g(x)} = \frac{2x}{x^2 - 2}$.

This is a first-order differential equation which we can solve for $f(x)$ using separation of variables or by realizing that it is a linear DE. (Our approach below is equivalent to separation of variables.)

Note that $\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x))$. Thus, integrating both sides of the DE,

$$\ln(f(x)) = \int \frac{1}{x - g(x)}\mathrm{d}x = \int \frac{2x}{x^2 - 2}\mathrm{d}x = \ln|x^2 - 2| + C.$$

We conclude that fixed-point iteration of $g(x)$ is equivalent to Newton's method applied to $f(x) = x^2 - 2$.

Comment. The general solution of the DE has one degree of freedom (the C above, which we chose as 0). On the other hand, we know from the beginning that Newton's method applied to $f(x)$ and $Df(x)$ results in the same fixed-point iteration.

(d) Newton's method applied to $f(x) = x^2 - a$ is equivalent to fixed-point iteration of $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$.

Comment. The resulting method for computing square roots $\sqrt{a}$ is known as the **Babylonian method**. It consists of starting with an approximation $x_0 \approx \sqrt{a}$ and then iteratively computing $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

https://en.wikipedia.org/wiki/Methods_of_computing_square_roots

Order of convergence

Example 58. Suppose that x_n converges to x^* in such a way that the number of correct digits doubles from one term to the next. What does that mean in terms of the error $e_n = |x_n - x^*|$?

Comment. This is roughly what we observed numerically for the Newton method in Example 41.

Comment. It doesn't matter which base we are using because the number of digits in one base is a fixed constant multiple of the number of digits in another base. Make sure that this clear! (If unsure, how does the number of digits of an integer x in base 2 relate to the number of digits of x in base 10?)

Solution. Recall that the number of correct digits in base b is about $-\log_b(e_n)$.
Doubling these from one term to the next means that $-\log_b(e_{n+1}) \approx -2\log_b(e_n)$.
Equivalently, $\log_b(e_{n+1}) - 2\log_b(e_n) = \log_b\left(\frac{e_{n+1}}{e_n^2}\right) \approx 0$.
This in turn is equivalent to $\frac{e_{n+1}}{e_n^2} \approx 1$.

What if the number of correct digits triples? By the above arguments, we would have $\frac{e_{n+1}}{e_n^3} \approx 1$.
Of course, there is nothing special about 2 or 3.

Example 59. Suppose that x_n converges to x^*. Let $e_n = |x_n - x^*|$ be the error and $d_n = -\log_b(e_n)$ be the number of correct digits (in base b). If $d_{n+1} = A d_n + B$, what does that mean in terms of the error e_n?

Solution. $-\log_b(e_{n+1}) = -A\log_b(e_n) + B$ is equivalent to $\log_b(e_{n+1}) - A\log_b(e_n) = \log_b\left(\frac{e_{n+1}}{e_n^A}\right) = -B$.
This in turn is equivalent to $\frac{e_{n+1}}{e_n^A} = b^{-B}$.

This motivates the following definition.

Definition 60. Suppose that x_n converges to x^*. Let $e_n = |x_n - x^*|$. We say that x_n **converges to x of order q and rate r** if

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^q} = r.$$

Order 1. Convergence of order 1 is called **linear convergence**. As in the previous example, the rate r provides information on the number of additional correct digits per term.

Order 2. Convergence of order 2 is also called **quadratic convergence**. As we saw above, it means that number of correct binary digits d_n roughly doubles from one term to the next. More precisely, $d_{n+1} \approx 2d_n + B$ where the rate $r = 2^{-B}$ tells us that $B = -\log_2(r)$. [Note that r has the advantage of being independent of the base in which we measure the number of correct digits.]