Review: Taylor series, continued

Review. If $f(x)$ is **analytic** around $x=c$, then it equals its **Taylor series** of $f(x)$ at $x=c$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \ldots$$

Example 47. Determine the Taylor series of $\ln x$ at $x=1$.

Solution. If $f(x)=\ln(x)$, then $f'(x)=\frac{1}{x}$, $f''(x)=-\frac{1}{x^2}$, $f'''(x)=2\frac{1}{x^3}$, $f^{(4)}(x)=2\cdot 3\frac{1}{x^4}$, ...

For $n\geqslant 1$, we thus have $f^{(n)}(x)=(-1)^{n+1}\frac{(n-1)!}{x^n}$ and, in particular, $\frac{f^{(n)}(1)}{n!}=(-1)^{n+1}\frac{(n-1)!}{n!}=\frac{(-1)^{n+1}}{n}$.

Consequently, we have $\ln x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$.

Comment. By replacing x with $1-x$, we obtain $\sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right)$.

If we take the derivative of both sides, we further find $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. This is the famous **geometric series**.

The truncation $\sum_{n=0}^{M}\frac{f^{(n)}(c)}{n!}(x-c)^n$ is called the ***M*th Taylor polynomial** of $f(x)$ at $x=c$.

Comment. The Mth Taylor polynomial is a polynomial of degree at most M (note that the degree can be smaller if $f^{(M)}(c)=0$).

Important comment. The first Taylor polynomial of $f(x)$ at $x=c$ is the tangent line of $f(x)$ at $x=c$. In other words, it is the best linear approximation of $f(x)$ at $x=c$.

Likewise, the Mth Taylor polynomial is the best polynomial approximation at $x=c$ of degree up to M.

We have the following fundamental result for what happens when we truncate a Taylor series.

Theorem 48. (Taylor's theorem with error term) Suppose that $f(x)$ is $M+1$ times continuously differentiable on the interval between x and c. Then we have

$$f(x) = \underbrace{\sum_{n=0}^{M}\frac{f^{(n)}(c)}{n!}(x-c)^n}_{M\text{th Taylor polynomial}} + \underbrace{\frac{f^{(M+1)}(\xi)}{(M+1)!}(x-c)^{M+1}}_{\text{error term}}$$

for some ξ between x and c.

Advanced comment. We only need that $f(x)$ is $M+1$ times differentiable and that $f^{(M)}(x)$ is continuous.

The special case $M=0$ of Taylor's theorem is equivalent to the mean value theorem:

Theorem 49. (mean value theorem) Suppose that $f(x)$ is differentiable on $[a,b]$. Then there exists $\xi\in(a,b)$ such that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Make a picture! □

Note that Taylor's theorem provides us with a representation for the error when we approximate $f(x)$ with a Taylor polynomial. This is illustrated in the next example.

Example 50. Suppose we use the approximation $e^x \approx 1 + x + \frac{x^2}{2}$.

(a) Using Taylor's theorem, provide an upper bound for the error on the interval $[0, 1]$.

(b) Using Taylor's theorem, provide an upper bound for the error on the interval $[0, 0.1]$.

(c) Using Taylor's theorem, how many terms of the Taylor series do we need so that the error on $[0, 0.1]$ is less than 10^{-16}?

Solution. Note that $1 + x + \frac{x^2}{2}$ is the 2nd Taylor polynomial of e^x at $x = 0$.

(a) Taylor's theorem implies that

$$e^x - \left(1 + x + \frac{x^2}{2}\right) = \frac{e^\xi}{3!} x^3$$

for some ξ between 0 and x.
Note that $|e^\xi| \leqslant e$ for all $\xi \in [0, 1]$.
On the other hand, $|x^3| \leqslant 1$ for all $x \in [0, 1]$.
We therefore conclude that the error on the interval $[0, 1]$ is bounded by

$$\left| e^x - \left(1 + x + \frac{x^2}{2}\right) \right| = \left| \frac{e^\xi}{3!} x^3 \right| \leqslant \frac{e}{3!} \approx 0.453.$$

Comment. In this simple case, we can determine the maximal error exactly (without using Taylor's theorem). Since the function $e^x - \left(1 + x + \frac{x^2}{2}\right)$ is increasing on the interval $[0, 1]$, starting with the value 0, the maximal error must occur at $x = 1$ and is $e - \frac{5}{2} \approx 0.218$. We thus find that our earlier error bound was a bit conservative but not a bad upper bound.

(b) As above, we conclude that the error on the interval $[0, 0.1]$ is bounded by

$$\left| e^x - \left(1 + x + \frac{x^2}{2}\right) \right| = \left| \frac{e^\xi}{3!} x^3 \right| \leqslant \frac{e^{0.1}}{3!} 0.1^3 \approx 0.000184 = 1.84 \cdot 10^{-4}.$$

Comment. For comparison, as above, the maximal actual error is $1.71 \cdot 10^{-4}$.

(c) By Taylor's theorem,

$$|e^x - p_M(x)| = \left| \frac{e^\xi}{(M+1)!} x^{M+1} \right| \leqslant \frac{e^{0.1}}{(M+1)!} 0.1^{M+1}.$$

We wish to choose M so that the right-hand side is less than 10^{-16}. Since the right-hand side decreases very rapidly, we simply increase M until that happens:

$$\frac{e^{0.1}}{3!} 0.1^3 \approx 1.8 \cdot 10^{-4}, \quad \ldots, \quad \frac{e^{0.1}}{9!} 0.1^9 \approx 3.0 \cdot 10^{-15}, \quad \frac{e^{0.1}}{10!} 0.1^{10} \approx 3.0 \cdot 10^{-17}.$$

We conclude that the 9th Taylor polynomial will approximate e^x in such a way that the error on $[0, 0.1]$ is less than 10^{-16}.

Fixed-point iteration

Definition 51. x^* is a **fixed point** of a function $f(x)$ if $f(x^*) = x^*$.

Example 52. Determine all fixed points of the function $f(x) = x^3$.

Solution. $x^3 = x$ has the three solutions $x^* = 0, \pm 1$ (and a cubic equation cannot have more than 3 solutions). These are the fixed points.

Idea. Suppose x^* is a fixed point of a continuous function f. If $x_n \approx x^*$, then $f(x_n) \approx f(x^*) = x^* \approx x_n$. If we can guarantee that $f(x_n)$ is closer to x^* than x_n, then we can set

$$x_{n+1} = f(x_n),$$

with the expectation that iterating this process will bring us closer and closer to x^*.

When does this converge? This process converges if $|f(x_n) - x^*| < |x_n - x^*|$ for all x_n close to x^*.

This condition is equivalent to $\left|\frac{f(x_n) - x^*}{x_n - x^*}\right| < 1$.

Since $x^* = f(x^*)$, we have $\frac{f(x_n) - x^*}{x_n - x^*} = \frac{f(x_n) - f(x^*)}{x_n - x^*} \approx f'(x^*)$ provided that x_n is sufficiently close to x^*.

This essentially proves the following result. (See below for a full proof using the mean value theorem.)

Theorem 53. Suppose that x^* is a fixed point of a continuously differentiable function f. If $|f'(x^*)| < 1$, then **fixed-point iteration**

$$x_{n+1} = f(x_n), \quad x_0 = \text{initial approximation},$$

converges to x^* locally.

In that case, we say that x^* is an **attracting fixed point**.

Divergence. If $|f'(x^*)| > 1$, then x^* is a **repelling fixed point**. Our argument shows that fixed-point iteration will not converge to x^* except in the "freak" case where $x_n \not\approx x^*$ but $f(x_n) = x^*$.

Comment. Local convergence means that we have convergence for all initial values x_0 close enough to x^*.

Proof. Note that

$$\begin{aligned} x_{n+1} - x^* &= g(x_n) - g(x^*) \\ &= g'(\xi_n)(x_n - x^*) \end{aligned}$$

where we applied the mean value theorem for the second equation and where ξ_n is between x_n and x^*. Thus

$$|x_{n+1} - x^*| = |g'(\xi_n)| \cdot |x_n - x^*|$$

Since g' is continuous and $|g'(x^*)| < 1$, we have $|g'(x)| < \delta$ for some $\delta < 1$ for all x sufficiently close to x^*. If x_0 is sufficiently to x^* in that sense, then it follows that $|x_1 - x^*| < \delta \cdot |x_0 - x^*|$. In particular, x_1 is even closer to x^* and we can repeat this argument to conclude that $|x_{n+1} - x^*| < \delta \cdot |x_n - x^*|$ for all n. This implies that $|x_n - x^*| < \delta^n \cdot |x_0 - x^*|$. Since $\delta < 1$, this further implies that x_n converges to x^*. □