

Example 35. `Python` We can easily adjust our code from Example 30 for the bisection method to handle the regula falsi method. We only need to change the line that previously computed the midpoint $c = \frac{a+b}{2}$ and replace it with $c = \frac{af(b) - bf(a)}{f(b) - f(a)}$ instead:

```
>>> def regulafalsi(f, a, b, nr_steps):
    for i in range(nr_steps):
        c = (a*f(b) - b*f(a)) / (f(b) - f(a))
        if f(a)*f(c) < 0:
            b = c
        else:
            a = c
    return [a, b]
```

Comment. This code is using 6 function evaluations per iteration. As we did in the case of the bisection method, rewrite the code to only use a single function evaluation per iteration.

Let us use this code to automatically perform the computations we did in Example 34. Again, we use fractions to get exact values for easier comparison.

```
>>> def my_f(x):
    return x**3 - 2

>>> from fractions import Fraction

>>> regulafalsi(my_f, Fraction(1), Fraction(2), 1)
[Fraction(8, 7), Fraction(2, 1)]

>>> regulafalsi(my_f, Fraction(1), Fraction(2), 2)
[Fraction(75, 62), Fraction(2, 1)]

>>> regulafalsi(my_f, Fraction(1), Fraction(2), 3)
[Fraction(37538, 30301), Fraction(2, 1)]

>>> regulafalsi(my_f, Fraction(1), Fraction(2), 4)
[Fraction(1534043307, 1226096954), Fraction(2, 1)]

>>> regulafalsi(my_f, Fraction(1), Fraction(2), 5)
[Fraction(15236748520786296242, 12128315482217382469), Fraction(2, 1)]
```

No wonder that we stopped after 3 iterations by hand...

Important comment. The final two outputs give us an indication why performance-critical scientific computations are usually done using floats even if all involved quantities could be exactly represented as rational numbers. Compute the next iteration! The numerator and denominator integers can no longer be stored as 64 bit integers. And this after a measly 6 iterations of a simple algorithm!

This is a typical problem with any exact expressions. In practice, their complexity (the number of bits required to store them) often grows too fast.

Example 36. In Example 34, which we continued in the previous example, only the left point ever got updated. Will that always be the case? Explain!

Solution. For $f(x) = x^3 - 2$, we have $f'(x) = 3x^2$ and $f''(x) = 6x$.

Therefore we have $f'(x) > 0$ as well as $f''(x) > 0$ for all $x \in [1, 2]$. This means that our function is increasing as well as concave up (on the interval $[1, 2]$).

Because it is concave up, its graph will always lie below the secant lines we construct (see below).

Combined with the function being increasing, the regula falsi points (roots of the secant lines) will always be to the left of the true root (here $\sqrt[3]{2}$).

Accordingly, the right endpoint will never get updated.

Review of concavity. Recall that a function $f(x)$ is **concave up** (like any part of a parabola opening upward) at $x = c$ if $f''(c) > 0$. At such a point, the graph of the function lies above the tangent line (at least locally). Make a sketch! On the other hand, this means that for sufficiently small intervals $[a, b]$ around c , the graph will lie below the secant line through $(a, f(a))$ and $(b, f(b))$.

Let us note the following differences between bisection and regula falsi:

- The intervals produced by bisection shrink by a factor of $1/2$ per iteration. On the other hand, the length intervals produced by regula falsi usually don't drop below a certain length.

For instance. This is illustrated by Example 34. In that case, the generated intervals are all of the form $[a, 2]$ where the left side a approaches $\sqrt[3]{2}$ from below. In particular, these intervals will always have length larger than $2 - \sqrt[3]{2} \approx 0.74$.
- Despite this, the sequence c_n of "new" interval endpoints produced by regula falsi can be shown to always converge to a root. Often the approximations c_n converge faster than the approximations obtained through bisection, but it can also be the other way around.

Comment. There are, however, variations of regula falsi that are more reliably faster than bisection.
- Bisection is guaranteed to converge to a root at a certain rate (namely, one bit per iteration). Regula falsi frequently but not always converges faster, but we cannot guarantee a certain rate (this depends on the involved function $f(x)$).

Example 37. Suppose we use bisection or regula falsi to compute a root of some function. Several iterations result in the intervals $[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$. Based on these intervals, what is our approximation of the root?

- In the case of bisection.
- In the case of regula falsi.

Solution.

- In the case of bisection, the generically best choice for our approximation is the midpoint of the final interval $(a_n + b_n)/2$.
- In the case of regula falsi, our approximation is the "new" endpoint of the final interval. More precisely, the approximation is a_n if $b_n = b_{n-1}$ and it is b_n if $a_n = a_{n-1}$.

Secant method

The **secant method** for computing a root of a function $f(x)$ is a modification of regula falsi where we do not try to bracket the root (in other words, we do not produce intervals that are guaranteed to contain the root).

Instead, starting with two initial approximations x_0 and x_1 , we construct x_2, x_3, \dots by the rule

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

Comment. In other words, if $a = x_{n-1}$ and $b = x_n$ are the two most recent approximations, then the next approximation is

$$x_{n+1} = c = \frac{a f(b) - b f(a)}{f(b) - f(a)},$$

and, as in regula falsi, this is the root of the secant line through $(a, f(a))$ and $(b, f(b))$. While regula falsi next determines whether to continue with the interval $[a, c]$ or with $[c, b]$, the secant method always continues with b and c as the next pair of approximations (in particular, the root does not need to lie between b and c).

Advanced comment. The formula for x_{n+1} is somewhat problematic because it is prone to round-off errors. Namely, if x_n converges to a root of $f(x)$, then in both the numerator and denominator of that formula we are subtracting numbers of almost equal value. This can result in damaging loss of precision.

Why is this not an issue for regula falsi? (Hint: What do we know about the signs of $f(a)$ and $f(b)$?)

Example 38. Determine an approximation for $\sqrt[3]{2}$ by applying the secant method to the function $f(x) = x^3 - 2$ with initial approximations $x_0 = 1$ and $x_1 = 2$. Perform 3 steps.

Solution.

- $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{8}{7}$
- $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{75}{62}$
- $x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{989,312}{782,041}$

After 3 steps of the secant method, our approximation for $\sqrt[3]{2}$ is $\frac{989,312}{782,041} \approx 1.265$.

Comment. For comparison, $\sqrt[3]{2} \approx 1.2599$.

Compare Example 38 to Example 34 where we used regula falsi instead (note how the first two iterations resulted in the same approximations). In operational terms, the secant method is a simpler version of regula falsi since we are not trying to determine an interval that is guaranteed to contain a root.

It may therefore come as a surprise that the secant method typically converges considerably faster than regula falsi. However, we no longer have a guarantee of convergence (and the situation in general depends on the initial approximations as well as the function $f(x)$).