Final: practice

MATH 436 — Numerical Analysis
final exam: Tuesday, Dec 6

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2. Use the trapezoidal rule to approximate $\int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x = \frac{\pi}{4}$.

(a) Use $h = \frac{1}{3}$ and $h = \frac{1}{6}$.

(b) Using Richardson extrapolation, combine the previous two approximations to obtain an approximation of higher order. What are absolute and relative error?

(c) The extrapolated approximation is equivalent to the outcome of which method applied with $h = \frac{1}{6}$?

Solution. Let us write $f(x) = \frac{1}{x^2+1}$.

(a) $h = \frac{1}{3}$: $\int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x \approx \frac{h}{2}\left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1)\right] = \frac{1}{6}\left[1 + 2\cdot\frac{9}{10} + 2\cdot\frac{9}{13} + \frac{1}{2}\right] = \frac{203}{260} \approx 0.7808$

$h = \frac{1}{6}$: $\int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x \approx \frac{h}{2}\left[f(0) + 2f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{3}\right) + 2f\left(\frac{5}{6}\right) + f(1)\right]$

$= \frac{1}{12}\left[1 + 2\cdot\frac{36}{37} + 2\cdot\frac{9}{10} + 2\cdot\frac{4}{5} + 2\cdot\frac{9}{13} + 2\cdot\frac{36}{61} + \frac{1}{2}\right] = \frac{2{,}761{,}249}{3{,}520{,}920} \approx 0.7842$

(b) Let us write $A(h)$ and $A\left(\frac{h}{2}\right)$ for our two approximations, and A^* for the true value of the integral.

Since $A(h)$ is an approximation of order 2, we expect $A(h) \approx A^* + Ch^2$ for some constant C.

Correspondingly, $A\left(\frac{h}{2}\right) \approx A^* + \frac{1}{4}Ch^2$. Hence, $4A\left(\frac{h}{2}\right) - A(h) \approx (4-1)A^* = 3A^*$.

Hence, the Richardson extrapolation is $R := \frac{1}{3}\left[4A\left(\frac{h}{2}\right) - A(h)\right] = \frac{1}{3}\left[4\cdot\frac{2{,}761{,}249}{3{,}520{,}920} - \frac{203}{260}\right] = \frac{829{,}597}{1{,}056{,}276} \approx 0.78539795$.

Since the exact value is $\frac{\pi}{4} \approx 0.78539816$, the absolute error is $\left|R - \frac{\pi}{4}\right| \approx 2.18\cdot 10^{-7}$ while the relative error is $\left|\left(R - \frac{\pi}{4}\right)/\left(\frac{\pi}{4}\right)\right| \approx 2.78\cdot 10^{-7}$.

(Of course, you will not have to calculate with numbers like the above by hand on the exam.)

(c) Simpson's rule

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Problem 3. Consider the initial value problem $y' = xy^2 + 1,\ y(1) = 0$.

(a) Approximate the solution $y(x)$ for $x \in [1, 3]$ using Euler's method with 3 steps. In particular, what is the approximation for $y(3)$?

(b) What is the order of the local truncation error? The global error?

(c) Spell out the Taylor method of order 3 for numerically solving this initial value problem.

Solution.

(a) The step size is $h = \frac{3-1}{3} = \frac{2}{3}$. We apply Euler's method with $f(x, y) = xy^2 + 1$:

$$\begin{array}{ll} x_0 = 1 & y_0 = 0 \\ x_1 = \frac{5}{3} & y_1 = y_0 + f(x_0, y_0)h = \frac{2}{3} \\ x_2 = \frac{7}{3} & y_2 = y_1 + f(x_1, y_1)h = \frac{2}{3} + \left(\frac{5}{3}\left(\frac{2}{3}\right)^2 + 1\right)\frac{2}{3} = \frac{148}{81} \\ x_3 = 3 & y_3 = y_2 + f(x_2, y_2)h = \frac{148}{81} + \left(\frac{7}{3}\left(\frac{148}{81}\right)^2 + 1\right)\frac{2}{3} = \frac{453914}{59049} \end{array}$$

In particular, the approximation for $y(3)$ is $y_3 = \frac{453914}{59049} \approx 7.68707$.

(b) The local truncation error is of order 2, and the global error is of order 1.

In other words, the local truncation error is $O(h^2)$ and the global error is $O(h)$, where h is the step size.

(c) The Taylor method of order 3 is based on the Taylor expansion

$$y(x + h) = y(x) + y'(x)h + \tfrac{1}{2}y''(x)h^2 + \frac{1}{6}y'''(x)h^3 + O(h^4),$$

where we have a local truncation error of $O(h^4)$ so that the global error will be $O(h^3)$.

From the DE we know that $y'(x) = xy^2 + 1$, which is $f(x, y)$. We differentiate this to obtain

$$y''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(xy^2 + 1) = y^2 + 2xyy' = y^2 + 2xy \cdot (xy^2 + 1) = y^2 + 2x^2y^3 + 2xy$$

which is $f'(x, y)$. We differentiate once more to find

$$\begin{aligned} y'''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(y^2 + 2x^2y^3 + 2xy) = 2yy' + 4xy^3 + 6x^2y^2y' + 2y + 2xy' \\ &= (2y + 6x^2y^2 + 2x)(xy^2 + 1) + 4xy^3 + 2y \\ &= 2(x + 2y + 4x^2y^2 + 3xy^3 + 3x^3y^4) \end{aligned}$$

which is $f''(x, y)$ (in the second step we replaced y' by $xy^2 + 1$).

Hence, the Taylor method of order 3 takes the form:

$$\begin{aligned} y_{n+1} &= y_n + f(x_n, y_n)h + \tfrac{1}{2}f'(x_n, y_n)h^2 + \tfrac{1}{6}f''(x_n, y_n)h^3 \\ &= y_n + (x_ny_n^2 + 1)h + \tfrac{1}{2}(y_n^2 + 2x_n^2y_n^3 + 2x_ny_n)h^2 + \tfrac{1}{3}(x_n + 2y_n + 4x_n^2y_n^2 + 3x_ny_n^3 + 3x_n^3y_n^4)h^3 \end{aligned}$$