Quiz #2

MATH 311 — Intro to Number Theory
Thursday, Oct 24

Please print your name:

Problem 1. (3 points) What is the last (decimal) digit of 7^{123456}?

Solution. We need to determine $7^{123456} \pmod{10}$. Since $\gcd(7,10)=1$ and $\phi(10)=\phi(2)\phi(5)=4$ and $123456 \equiv 56 \equiv 0 \pmod 4$, we have $7^{123456} \equiv 7^0 \equiv 1 \pmod{10}$. This means that the last (decimal) digit of 7^{123456} is 1. ☐

Problem 2. (9 points)

(a) Among the numbers $1, 2, \ldots, 54$, how many are coprime to 54? ☐

(b) If $n = p^2 q$, for distinct primes p, q, then $\phi(n) =$ ☐.

(c) How many solutions does the congruence $x^2 \equiv 4 \pmod{105}$ have? ☐

(d) How many solutions does the congruence $x^2 \equiv 4 \pmod{210}$ have? ☐

(e) How many solutions does the congruence $x^2 \equiv 4 \pmod{3135}$ have? ☐ $(3135 = 3 \cdot 5 \cdot 11 \cdot 19)$

(f) The multiplicative order of $3 \pmod{11}$ is ☐.

(g) The primitive roots modulo 7 are ☐.

(h) If $x \pmod n$ has multiplicative order k, then x^{2019} has multiplicative order ☐.

(i) What is the number of invertible residues modulo 75? ☐

Solution.

(a) $\phi(54) = \phi(2)\phi(27) = 27 - 9 = 18$

(b) If $n = p^2 q$, for distinct primes p, q, then $\phi(n) = \phi(p^2)\phi(q) = (p^2 - p)(q - 1)$.

(c) By the CRT, since $105 = 3 \cdot 5 \cdot 7$, the congruence has $2 \cdot 2 \cdot 2 = 8$ solutions.

(d) By the CRT, since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, the congruence has $1 \cdot 2 \cdot 2 \cdot 2 = 8$ solutions. (Note that $x^2 \equiv 4 \pmod 2$ only has one solution; namely, $x \equiv 0$.)

(e) By the CRT, since $3135 = 3 \cdot 5 \cdot 11 \cdot 19$, the congruence has $2 \cdot 2 \cdot 2 \cdot 2 = 16$ solutions.

(f) The multiplicative order of $3 \pmod{11}$ is 5.

(g) The primitive roots modulo 7 are $3, 5$.

(h) If $x \pmod n$ has multiplicative order k, then x^{2019} has multiplicative order $\frac{k}{\gcd(k, 2019)}$.

(i) $\phi(75) = \phi(3)\phi(25) = 2 \cdot 20 = 40$ ☐

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