19 Basic proof techniques

19.1 Proofs by contradiction

Example 174. (again) $\sqrt{5}$ is not rational.

Proof. Assume (for contradiction) that we can write $\sqrt{5} = \frac{n}{m}$ with $n, m \in \mathbb{N}$. By canceling common factors, we can ensure that this fraction is reduced.

Then $5m^2 = n^2$, from which we conclude that n is divisible by 5. Write $n = 5k$ for some $k \in \mathbb{N}$. Then $5m^2 = (5k)^2$ implies that $m^2 = 5k^2$. Hence, m is also divisible by 5. This contradicts the fact that the fraction n / m is reduced. Hence, our initial assumption must have been wrong. □

Variations. Does the same proof apply to, say, $\sqrt{7}$?

Which step of the proof fails for $\sqrt{9}$?

Comment. We showed earlier that $[1; 1, 1, 1, \ldots] = \frac{1+\sqrt{5}}{2}$. Since this is an infinite continued fraction, this proves that $\frac{1+\sqrt{5}}{2}$ is irrational. Consequently, $\sqrt{5}$ is irrational as well.

19.2 A famous example of a direct proof

Example 175. (Gauss) $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

Proof. Write $s(n) = 1 + 2 + \ldots + n$.

$2s(n) = (1 + 2 + \ldots + n) + (n + (n-1) + \ldots + 1) = (1+n) + (2+n-1) + \ldots + (n+1) = n \cdot (n+1)$. Done! □

Anecdote. 9 year old Gauss (1777-1855) and his classmates were tasked to add the numbers 1 to 100 (and not bother their teacher while doing so). Gauss was not writing much on his slate... just the final answer: 5050.

19.3 Proofs by induction

(induction) To prove that $\mathrm{CLAIM}(n)$ is true for all integers $n \geqslant n_0$, it suffices to show:

- **(base case)** $\mathrm{CLAIM}(n_0)$ is true.
- **(induction step)** If $\mathrm{CLAIM}(n)$ is true for some n, then $\mathrm{CLAIM}(n+1)$ is true as well.

Why does this work? By the base case, $\mathrm{CLAIM}(n_0)$ is true. Thus, by the induction step, $\mathrm{CLAIM}(n_0+1)$ is true. Applying the induction step again shows that $\mathrm{CLAIM}(n_0+2)$ is true, ...

Comment. In the induction step, we may even assume that $\mathrm{CLAIM}(n_0), \mathrm{CLAIM}(n_0+1), \ldots, \mathrm{CLAIM}(n)$ are all true. This is sometimes referred to as **strong induction**.

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Example 176. (Gauss, again) For all integers $n \geqslant 1$, $1+2+\ldots+n = \frac{n(n+1)}{2}$.

Proof. Again, write $s(n) = 1+2+\ldots+n$.
$\mathrm{CLAIM}(n)$ is that $s(n) = \frac{n(n+1)}{2}$.

- **(base case)** $\mathrm{CLAIM}(1)$ is that $s(1) = \frac{1(1+1)}{2} = 1$. That's true.
- **(induction step)** Assume that $\mathrm{CLAIM}(n)$ is true (the **induction hypothesis**) for some fixed n.

$$s(n+1) = s(n) + (n+1) = \underbrace{\frac{n(n+1)}{2}}_{\substack{\text{this is where we use}\\ \text{the induction hypothesis}}} + (n+1) = \frac{(n+1)(n+2)}{2}$$

This shows that $\mathrm{CLAIM}(n+1)$ is true as well.

By induction, the formula is therefore true for all integers $n \geqslant 1$. □

Comment. The claim is also true for $n=0$ (if we interpret the left-hand side correctly).

Example 177. Induction is not only a proof technique but also a common way to define things.

- The **factorial** $n!$ can be defined inductively (i.e. recursively) by

$$0! = 1, \quad (n+1)! = n! \cdot (n+1).$$

Comment. This may not seem impressive, because we can "spell out" $n! = 1 \cdot 2 \cdot 3 \cdots (n-1)n$ directly.

- The **Fibonacci numbers** F_n are defined inductively (i.e. recursively) by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}.$$

Getting a feeling. $F_2 = F_1 + F_0 = 1$, $F_3 = F_2 + F_1 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, ...

Comment. Though not at all obvious, there is a way to compute F_n directly. Let $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$. Then $F_n = \lfloor \varphi^n / \sqrt{5} \rceil$. Try it! For instance, $\varphi^{10}/\sqrt{5} \approx 55.0036$. That seems like magic at first. But it is the beginning of a general theory (look up, for instance, Binet's formula and C-finite sequences). Also, recall that we observed that F_{n+1}/F_n are the convergents of the continued fraction for φ.

Example 178. We are interested in the sums $s(n) = 1+2+4+\ldots+2^n$.

Getting a feeling. $s(1) = 1+2 = 3$, $s(2) = 1+2+4 = 7$, $s(3) = 1+2+4+8 = 15$, $s(4) = 31$

Conjecture. $s(n) = 2^{n+1} - 1$.

Proof by induction. The statement we want to prove by induction is: $s(n) = 2^{n+1} - 1$ for all integers $n \geqslant 1$.

- **(base case)** $s(1) = 1 = 2^{1+1} - 1$ verifies that the claim is true for $n=1$.
- **(induction step)** Assume that $s(n) = 2^{n+1} - 1$ is true for some fixed n.
  We need to show that $s(n+1) = 2^{n+2} - 1$.
  Using the induction hypothesis, $s(n+1) = s(n) + 2^{n+1} \overset{\mathrm{IH}}{=} (2^{n+1} - 1) + 2^{n+1} = 2^{n+2} - 1$. QED!

Direct proof. $2s(n) = 2(1+2+4+\ldots+2^n) = 2+4+\ldots+2^{n+1} = s(n) - 1 + 2^{n+1}$. Hence, $s(n) = 2^{n+1} - 1$.