Example 129.

(a) Show that 7 is a primitive root modulo 26.

(b) Using the first part, make a complete list of all primitive roots modulo 26.

Solution.

(a) We need to show that 7 has order $\phi(26) = 12$.

The order of 7 (or any invertible residue) must divide $\phi(26) = 12$. Hence, the only possibilities for orders are $1, 2, 3, 4, 6, 12$. The fact that $7^4 \equiv (-3)^2 \equiv 9 \not\equiv 1 \pmod{26}$ and $7^6 \equiv (-3)^3 \equiv -1 \not\equiv 1 \pmod{26}$ is enough (why?!) to conclude that the order of 7 must be 12.

(b) Since 7 is a primitive root, all other invertible residues are of the form 7^a.

Recall that 7^a has order $\frac{12}{\gcd(12, a)}$. Thus, 7^a is a primitive root if and only if $\gcd(12, a) = 1$.

Therefore, a list of all primitive roots modulo 26 is: $7, 7^5, 7^7, 7^{11}$

[These are $\phi(\phi(26)) = \phi(12) = 4$ many primitive roots.]

The same logic applies whenever there is at least one primitive root:

Theorem 130. (number of primitive roots) Suppose there is a primitive root modulo n. Then there are $\phi(\phi(n))$ primitive roots modulo n.

Proof. Let x be a primitive root. It has order $\phi(n)$. All other invertible residues are of the form x^a.

Recall that x^a has order $\frac{\phi(n)}{\gcd(\phi(n), a)}$. This is $\phi(n)$ if and only if $\gcd(\phi(n), a) = 1$. There are $\phi(\phi(n))$ values a among $1, 2, \ldots, \phi(n)$, which are coprime to $\phi(n)$.

In conclusion, there are $\phi(\phi(n))$ primitive roots modulo n. □

Comment. Recall that, for instance, there is no primitive root modulo 8. That's why we needed the assumption that there should be a primitive root modulo n (which is the case if and only if n is of the form $1, 2, 4, p^k, 2p^k$ for some odd prime p).

Corollary 131. There are $\phi(\phi(p)) = \phi(p-1)$ primitive roots modulo a prime p.

Example 132. Let p be an odd prime. Show that at most half of the invertible residues modulo p are primitive roots.

Solution. In other words, we need to show that $\frac{\phi(p-1)}{p-1} \leqslant \frac{1}{2}$. Let $p_1, p_2, \ldots$ be the primes, in increasing order, dividing $p-1$. Since $p \neq 2$, $p-1$ is divisible by 2, so that $p_1 = 2$.

Then, $\phi(p-1) = (p-1)\underbrace{\left(1 - \frac{1}{p_1}\right)}_{=1/2}\underbrace{\left(1 - \frac{1}{p_2}\right)\cdots}_{\leqslant 1} \leqslant \frac{1}{2}(p-1)$.

Consequently, $\frac{\phi(p-1)}{p-1} \leqslant \frac{\frac{1}{2}(p-1)}{p-1} = \frac{1}{2}$, as claimed.

In fact. Note that $\left(1 - \frac{1}{p_2}\right) < 1$ if there is a second prime. Our proof therefore actually shows that $\frac{\phi(p-1)}{p-1} = \frac{1}{2}$ if and only if $p-1$ is of the form 2^n (i.e. the only prime dividing $p-1$ is 2). Equivalently, if p is of the form $2^n + 1$.

Comment. Primes of the form $2^n + 1$ are known as **Fermat primes**. It can be shown that such a prime is, in fact, necessarily of the form $F_k = 2^{2^k} + 1$. The first five numbers $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$ are prime, and Fermat conjectured that F_k is prime for all $k \geqslant 0$. This was proven wrong by Euler who demonstrated that $F_5 = 2^{32} + 1 = 641 \cdot 6700417$ (this was way before the time, we could ask a computer to factor not-too-large numbers). To this day, it is not known whether any further Fermat primes exist.

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Example 133. Recall that, for every prime p, primitive roots exist. The total number of primitive roots is $\phi(\phi(p)) = \phi(p-1)$. The following computations in Sage indicate that typically a "decent" proportion (25-50%) of all invertible residues are primitive roots. The exact proportion is, of course $\frac{\phi(p-1)}{p-1}$ but to say more about the magnitude, we need the factorization of $p-1$.

Advanced comment. However, the number of primitive roots can (though this is very rare) be an arbitrarily small proportion. In fact, a result of Kátai shows that, for any $x \in [0,1]$, there is a proportion $P(x)$ of primes with $\frac{\phi(p-1)}{p-1} \leqslant x$, and that $P(x)$ is a strictly increasing continuous function with $P(0) = 0$ and $P(1/2) = 1$.

```
Sage] prime_range(30)
```

$[2, 3, 5, 7, 11, 13, 17, 19, 23, 29]$

```
Sage] euler_phi(26)
```

12

```
Sage] [p^2 for p in prime_range(30)]
```

$[4, 9, 25, 49, 121, 169, 289, 361, 529, 841]$

```
Sage] [euler_phi(p-1)/(p-1) for p in prime_range(30)]
```

$$\left[1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{5}{11}, \frac{3}{7}\right]$$

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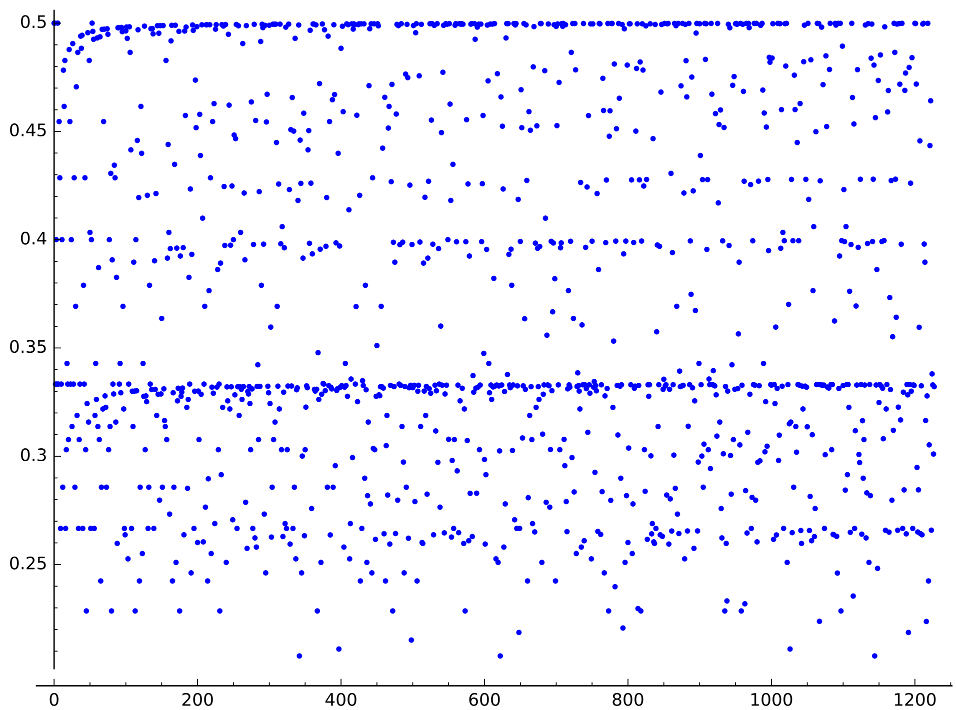
Sage] list_plot([euler_phi(p-1)/(p-1) for p in prime_range(3,10000)])
```



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