Review. continued fractions, convergents

Example 145. Determine the first few digits of the simple continued fraction of e.

Solution. $e = \boxed{2}.71828182846\ldots$

$e = 2 + \frac{1}{1/0.7182\ldots} = [2; a_1, a_2, \ldots]$ where $[a_1; a_2, \ldots] = 1/0.7182\ldots = \boxed{1}.3922\ldots$

$1/0.3922\ldots = \boxed{2}.5496\ldots$, $1/0.5496\ldots = \boxed{1}.8194\ldots$, $1/0.8194\ldots = \boxed{1}.2205\ldots$, $1/0.2205\ldots = \boxed{4}.5356\ldots$

Hence, $e = [2; 1, 2, 1, 1, 4, \ldots]$.

Computing more digits, we find $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ and the pattern continues.

Note. Assuming that the pattern does continue, this proves that e is irrational!

Example 146.

(a) Evaluate the first 4 convergents of $[2; 3, 2, 3, 2, \ldots]$ (and then, using the next result, compute 3 more convergents).

(b) Which number is represented by $[2; 3, 2, 3, 2, \ldots]$?

Solution.

(a) $C_0 = 2$

$C_1 = [2; 3] = 2 + \frac{1}{3} = \frac{7}{3} \approx 2.333$

$C_2 = [2; 3, 2] = 2 + \frac{1}{3 + \frac{1}{2}} = 2 + \frac{2}{7} = \frac{16}{7} \approx 2.286$

$C_3 = [2; 3, 2, 3] = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3}}} = \frac{55}{24} \approx 2.292$

Using the next result, we compute the convergents $C_n = \frac{p_n}{q_n}$ as follows:

n	-2	-1	0	1	2	3	4	5	6
a_n			2	3	2	3	2	3	2
p_n	0	1	2	7	16	55	126	433	992
q_n	1	0	1	3	7	24	55	189	433
C_n			$\frac{2}{1}$	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{55}{24}$	$\frac{126}{55}$	$\frac{433}{189}$	$\frac{992}{433}$

(b) Write $x = [2; 3, 2, 3, 2, \ldots]$. Then, $x = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \ldots}}} = 2 + \frac{1}{3 + \frac{1}{x}}$.

The equation $x = 2 + \frac{1}{3 + \frac{1}{x}}$ simplifies to $x - 2 = \frac{x}{3x + 1}$.

Further (note that, clearly $x \neq -\frac{1}{3}$ so that $3x + 1 \neq 0$) simplifies to $(x - 2)(3x + 1) = x$ or $3x^2 - 6x - 2 = 0$, which has the solutions $x = \frac{6 \pm \sqrt{36 + 24}}{6} = 1 \pm \sqrt{\frac{5}{3}}$.

Since $1 + \sqrt{\frac{5}{3}} \approx 2.291$ and $1 - \sqrt{\frac{5}{3}} \approx -0.291$, we conclude that $[2; 3, 2, 3, 2, \ldots] = 1 + \sqrt{\frac{5}{3}}$.

Advanced comment. The fractions $\frac{p_n}{q_n}$ are always reduced! Can you see how to conclude that $\gcd(p_n, q_n) = 1$ from the relation $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ (which can be proved by induction)?

We can see this relation quite nicely in the above table because $p_n q_{n-1} - p_{n-1} q_n$ is a 2×2 determinant taken from the rows containing p_n and q_n:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix} = -1, \quad \begin{vmatrix} 7 & 16 \\ 3 & 7 \end{vmatrix} = 1, \quad \begin{vmatrix} 16 & 55 \\ 7 & 24 \end{vmatrix} = -1, \quad \ldots$$

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Theorem 147. The kth convergent of the continued fraction $[a_0; a_1, a_2, \ldots]$ is

$$C_k = \frac{p_k}{q_k},$$

where p_k and q_k are characterized by

$$\begin{array}{l} p_k = a_k p_{k-1} + p_{k-2} \\ \text{with } p_{-2} = 0, \quad p_{-1} = 1 \end{array} \quad \text{and} \quad \begin{array}{l} q_k = a_k q_{k-1} + q_{k-2} \\ \text{with } q_{-2} = 1, \quad q_{-1} = 0 \end{array}.$$

Proof. We will prove the claim by induction on k. (More on that technique next time!)
First, we check the two base cases $k=0$, $k=1$ directly: $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$. In other words, $p_0 = a_0$, $q_0 = 1$ and $p_1 = a_0 a_1 + 1$, $q_1 = a_1$. This matches with the values from the recursion.
Next, we assume that the theorem is true for $k = 0, 1, \ldots, n$. In particular,

$$C_n = [a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}},$$

for any values $a_0, a_1, \ldots, a_n$. Note that $C_{n+1} = [a_0; a_1, a_2, \ldots, a_n, a_{n+1}] = \left[a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}\right]$. Replacing a_n with $a_n + \frac{1}{a_{n+1}}$, we therefore obtain

$$\begin{aligned} C_{n+1} = \left[a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}\right] &= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \\ &= \frac{(a_n a_{n+1} + 1) p_{n-1} + a_{n+1} p_{n-2}}{(a_n a_{n+1} + 1) q_{n-1} + a_{n+1} q_{n-2}} \\ &= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} \\ &= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}. \end{aligned}$$

The claim now follows by induction. □

Example 148. Determine $[1; 1, 1, 1, \ldots]$ as well as its first 6 convergents.

Solution. The first few convergents are $C_0 = 1$, $C_1 = [1; 1] = 2$, $C_2 = [1; 1, 1] = 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$.

Since this starts getting tedious, we instead compute the convergents $C_n = \frac{p_n}{q_n}$ recursively:

n	-2	-1	0	1	2	3	4	5	6
a_n			1	1	1	1	1	1	1
p_n	0	1	1	2	3	5	8	13	21
q_n	1	0	1	1	2	3	5	8	12
C_n			1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{12}$

Note that the C_n are quotients of Fibonacci numbers ($F_0 = 0, F_1 = 1, F_2 = 1, \ldots$)! To be precise, $C_n = \frac{F_{n+2}}{F_{n+1}}$.
Next, let's determine $x = [1; 1, 1, 1, \ldots]$ by observing that $x = 1 + \frac{1}{1 + \frac{1}{1 + \ldots}} = 1 + \frac{1}{x}$.
The equation $x = 1 + \frac{1}{x}$ simplifies to $x^2 - x - 1 = 0$, which has the solutions $x = \frac{1 \pm \sqrt{5}}{2}$.
Since $\frac{1 - \sqrt{5}}{2}$ is negative (while x is between $C_0 = 1$ and $C_1 = 2$), we conclude $[1; 1, 1, 1, \ldots] = \frac{1 + \sqrt{5}}{2} \approx 1.618$.
This is the **golden ratio** φ.
Comment. Note that we have shown, in particular, $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi \approx 1.618$.
Comment. As noticed in the previous example time, the fractions $\frac{p_n}{q_n} = \frac{F_{n+2}}{F_{n+1}}$ are always reduced. In other words, $\gcd(F_n, F_{n+1}) = 1$. Moreover, $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ implies that $F_n^2 - F_{n-1} F_{n+1} = (-1)^{n+1}$.

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Example 149. Determine the first few digits of the simple continued fraction of π, as well as the first few convergents.

Solution. $\pi = \boxed{3}.14159265359\ldots$

Computing more digits, we find $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, \ldots]$.

Since π is irrational, this is an infinite continued fraction. No pattern in this fraction is known.

We compute the convergents $C_n = \frac{p_n}{q_n}$ as follows:

n	-2	-1	0	1	2	3	4	5	6
a_n			3	7	15	1	292	1	1
p_n	0	1	3	22	333	355	$103,993$	$\ldots$	$\ldots$
q_n	1	0	1	7	106	113	$33,102$	$\ldots$	$\ldots$
C_n			3	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103,993}{33,102}$	$\ldots$	$\ldots$

Comment. For $n \geqslant 1$, each approximation $x \approx \frac{p_n}{q_n}$ is best possible in the sense that it is better than any other approximation $\frac{a}{b}$ with $b \leqslant q_n$. In other words, if $\left|x - \frac{a}{b}\right| < \left|x - \frac{p_n}{q_n}\right|$, then $b > q_n$.

Comment. Because of this, it is natural to expect that the approximations $\frac{22}{7}$ and $\frac{355}{113}$ are particularly good, because they are followed by much "bigger" fractions.

Indeed, $\frac{22}{7} = \boxed{3.14}28\ldots$ and $\frac{355}{113} = \boxed{3.141592}92\ldots$ are very good approximations to π.

Comment. It is known that π is irrational, so that the above "wild" continued fraction will go on forever.

Embarrassingly, we do not know whether, for instance, $e + \pi = 5.85987448205\ldots$ is irrational.

$e + \pi = [5; 1, 6, 7, 3, 21, 2, 1, 2, 2, 1, 1, 2, 3, 3, 2, 5, 2, 1, 1, \ldots]$

All evidence points to it being irrational, but nobody has a proof. (In particular, we cannot be sure that this continued fraction goes on forever.)

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