Final: practice

MATH 311 — Intro to Number Theory
final exam: Tuesday, Dec 11

Please print your name:

As usual, calculators will not be permitted on the final exam. The numbers on the exam will be suitable for calculating by hand.

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams. (The exams with and without solutions are posted.)

(c) Do the problems below. (Solutions are posted.)

Problem 2.

(a) State Wilson's theorem.

(b) List all quadratic residues modulo 21.

(c) What is the number of invertible quadratic residues modulo 101?

(d) What is the number of invertible quadratic residues modulo 77?

(e) Which real numbers have a finite continued fraction?

Solution.

(a) If p is a prime, then $(p-1)! \equiv -1 \pmod{p}$.

(b) $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 9$, $(\pm 4)^2 = 16$, $(\pm 5)^2 \equiv 4$, $(\pm 6)^2 \equiv 15$, $(\pm 7)^2 \equiv 7$, $(\pm 8)^2 \equiv 1$, $(\pm 9)^2 \equiv 18$, $(\pm 10)^2 \equiv 16$

In summary, the quadratic residues are $0, 1, 4, 7, 9, 15, 16, 18$.

(The invertible quadratic residues are $1, 4, 16$. That's $\phi(21)/4 = \frac{\phi(3)\phi(7)}{4} = 3$ many.)

(c) Since 101 is a prime, there is $\frac{\phi(101)}{2} = 50$ invertible quadratic residues modulo 101.

(d) Since $77 = 7 \cdot 11$, there is $\frac{\phi(77)}{4} = 15$ invertible quadratic residues modulo 77.

(e) These are precisely the rational numbers. □

Problem 3.

(a) Which number is represented by the continued fraction $[1; 2, 1, 2, 1, 2]$?

Armin Straub
straub@southalabama.edu

(b) Determine all convergents of $[1;2,1,2,1,2]$.

(c) Which number is represented by the infinite continued fraction $[1;2,1,2,1,2,1,2,\ldots]$?

(d) Compare, numerically, the first six convergents (computed above) to the value of the infinite continued fraction.

Solution.

(a) $[1;2,1,2,1,2] = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}}}} = \frac{41}{30}$

Comment. Of course, we can simplify this continued fraction directly. But that is a bit time consuming and prone to errors. A better is way is to compute the convergents recursively as we do in the next part.

(b) The convergents are $C_0 = 1$, $C_1 = [1;2] = 1 + \frac{1}{2} = \frac{3}{2}$, $C_2 = [1;2,1] = 1 + \frac{1}{2 + \frac{1}{1}} = \frac{4}{3}$.

We can continue like that but the computations will get more involved. Instead, we should proceed recursively. Recall from class that the convergents $C_n = \frac{p_n}{q_n}$ of $[a_0; a_1, a_2, \ldots]$ are characterized by

$$\begin{array}{l} p_k = a_k p_{k-1} + p_{k-2} \\ \text{with } p_{-2} = 0, \quad p_{-1} = 1 \end{array} \qquad \text{and} \qquad \begin{array}{l} q_k = a_k q_{k-1} + q_{k-2} \\ \text{with } q_{-2} = 1, \quad q_{-1} = 0 \end{array}.$$

The corresponding calculations of p_n and q_n are as follows:

n	-2	-1	0	1	2	3	4	5
a_n			1	2	1	2	1	2
p_n	0	1	1	3	4	11	15	41
q_n	1	0	1	2	3	8	11	30
C_n			1	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{11}{8}$	$\frac{15}{11}$	$\frac{41}{30}$

(c) Write $x = [1;2,1,2,1,2,1,2,\ldots]$. Then, $x = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \ldots}}}} = 1 + \frac{1}{2 + \frac{1}{x}}$.

The equation $x = 1 + \frac{1}{2 + \frac{1}{x}}$ simplifies to $x - 1 = \frac{x}{2x+1}$. Further (note that, clearly $x \neq -\frac{1}{2}$ so that $2x + 1 \neq 0$) simplifies to $(x-1)(2x+1) = x$ or $2x^2 - 2x - 1 = 0$, which has the solutions $x = \frac{2 \pm \sqrt{4+8}}{4} = \frac{1 \pm \sqrt{3}}{2}$.

Since $\frac{1+\sqrt{3}}{2} \approx 1.366$ and $\frac{1-\sqrt{3}}{2} \approx -0.366$, we conclude that $[1;2,1,2,1,2,1,2,\ldots] = \frac{1+\sqrt{3}}{2}$.

(d) $C_0 = 1$, $C_1 = \frac{3}{2} = 1.5$, $C_2 = \frac{4}{3} \approx 1.333$, $C_3 = \frac{11}{8} = 1.375$, $C_4 = \frac{15}{11} \approx 1.364$, $C_5 = \frac{41}{30} \approx 1.367$

These values quickly approach $\frac{1+\sqrt{3}}{2} \approx 1.366$ in the expected alternating fashion. □

Problem 4.

(a) Express the numbers $\frac{252}{193}$ and $-\frac{337}{221}$ as a simple continued fraction.

(b) Is this the unique simple continued fraction representing $\frac{252}{193}$? Explain!

Solution.

(a) The simplest way to obtain the continued fraction for $\frac{252}{193}$ is via the Euclidean algorithm:

$252 = \boxed{1} \cdot 193 + 59, \quad 193 = \boxed{3} \cdot 59 + 16, \quad 59 = \boxed{3} \cdot 16 + 11, \quad 16 = \boxed{1} \cdot 11 + 5, \quad 11 = \boxed{2} \cdot 5 + 1, \quad 5 = \boxed{5} \cdot 1 + 0$

Armin Straub
straub@southalabama.edu

Hence, $\frac{252}{193} = [1; 3, 3, 1, 2, 5]$.

To determine a simple continued fraction for $-\frac{337}{221}$, we first write $-\frac{337}{221} = -2 + \frac{105}{221} = \boxed{-2} + \frac{1}{\frac{221}{105}}$. We then proceed using the Euclidean algorithm applied to $\frac{221}{105}$.

$$221 = \boxed{2} \cdot 105 + 11, \quad 105 = \boxed{9} \cdot 11 + 6, \quad 11 = \boxed{1} \cdot 6 + 5, \quad 6 = \boxed{1} \cdot 5 + 1, \quad 5 = \boxed{5} \cdot 1 + 0.$$

Combined, $-\frac{337}{221} = [-2; 2, 9, 1, 1, 5]$.

(b) No, a finite continued fraction can always be expressed in two ways because of the simple relation $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$, assuming $a_n > 1$. In this case, we also have $\frac{252}{193} = [1; 3, 3, 1, 2, 4, 1]$. □

Armin Straub
straub@southalabama.edu