Example 134. Determine $[1;1,1,1,\ldots]$ as well as its first 6 convergents.

Solution. The first few convergents are $C_0=1$, $C_1=[1;1]=2$, $C_2=[1;1,1]=1+\frac{1}{1+\frac{1}{1}}=\frac{3}{2}$.

Since this starts getting tedious, we instead compute the convergents $C_n=\frac{p_n}{q_n}$ recursively:

n	-2	-1	0	1	2	3	4	5	6
a_n			1	1	1	1	1	1	1
p_n	0	1	1	2	3	5	8	13	21
q_n	1	0	1	1	2	3	5	8	12
C_n			1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{12}$

Note that the C_n are quotients of Fibonacci numbers! To be precise, $C_n=\frac{F_{n+2}}{F_{n+1}}$.

Next, let's determine $x=[1;1,1,1,\ldots]$. Then, $x=1+\frac{1}{1+\frac{1}{1+\ldots}}=1+\frac{1}{x}$.

The equation $x=1+\frac{1}{x}$ simplifies to $x^2-x-1=0$, which has the solutions $x=\frac{1\pm\sqrt{5}}{2}$.

Since $\frac{1-\sqrt{5}}{2}$ is negative (while x is between $C_0=1$ and $C_1=2$), we conclude $[1;1,1,1,\ldots]=\frac{1+\sqrt{5}}{2}\approx 1.618$.

This is the **golden ratio** φ.

Comment. Early in the semester, we numerically observed that $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\varphi\approx 1.618$. Now we know!

Comment. As noticed last time, the fractions $\frac{p_n}{q_n}=\frac{F_{n+2}}{F_{n+1}}$ are always reduced. In other words, this means $\gcd(F_n,F_{n+1})=1$ (recall that we derived this directly in one homework problem).

Moreover, $p_nq_{n-1}-p_{n-1}q_n=(-1)^n$ implies that $F_n^2-F_{n-1}F_{n+1}=(-1)^{n+1}$.

Theorem 135. (representing a real number as a simple continued fraction)

- An irrational number x has a unique representation as a simple continued fraction. This continued fraction is infinite.
- A rational number x has exactly two representations as a simple continued fraction. Both are finite (one ends in a 1 and the other doesn't).

Proof. Let x be a positive real number. Let us think about how a continued fraction for x has to look like. [The argument for negative x is essentially the same. For negative x, a_0 will be negative but the remainder and the other digits are positive.]

By Theorem 129, we have $C_0\leqslant x\leqslant C_1$ where $C_0=a_0$ and $C_1=a_0+\frac{1}{a_1}\leqslant a_0+1$.

Hence, $a_0\leqslant x\leqslant a_0+1$ which means that a_0 has to be the integer $a_0=\lfloor x\rfloor$.

(unless) Well, unless x is an integer itself, in which case we have the two possibilities $a_0=x$ or $a_0=x-1$. But in that special case, we are done: the continued fraction for x is finite and there is exactly the two representations $x=[x]$ and $x=[x-1;1]$.

So, $x=a_0+\frac{1}{y}$ with $y=\frac{1}{x-a_0}>0$, and the continued fraction for x is $x=[a_0;b_0,b_1,\ldots]$ if $y=[b_0;b_1,\ldots]$. We now repeat our argument, starting with the positive real number y (so that $b_0=\lfloor y\rfloor$).

There is two possibilities:

- The process stops along the way because the number we are looking at is an integer (the "unless" case). In that case, we get exactly two finite simple continued fractions for x (one of which ends in 1).

  This happens if and only if x is rational (from the Euclidian algorithm we know that every rational number has a finite simple continued fraction; conversely, a finite simple continued fraction necessarily represents a rational number).
- The process continues indefinitely. In that case, we get a (unique) infinite simple continued fraction for x. □

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Example 136. Determine the first few digits of the simple continued fraction of e.

Solution. $e = \boxed{2}.71828182846...$

$e = 2 + \frac{1}{1/0.7182...} = [2; a_1, a_2, ...]$ where $[a_1; a_2, ...] = 1/0.7182... = \boxed{1}.3922...$

$1/0.3922... = \boxed{2}.5496...$, $1/0.5496... = \boxed{1}.8194...$, $1/0.8194... = \boxed{1}.2205...$, $1/0.2205... = \boxed{4}.5356...$

Hence, $e = [2; 1, 2, 1, 1, 4, ...]$.

Computing more digits, we find $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$ and the pattern continues.

Note. Assuming that the pattern does continue, this proves that e is irrational!

Example 137. Determine the first few digits of the simple continued fraction of π, as well as the first few convergents.

Solution. $\pi = \boxed{3}.14159265359...$

Computing more digits, we find $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, ...]$.

Since π is irrational, this is an infinite continued fraction. No pattern in this fraction is known.

We compute the convergents $C_n = \frac{p_n}{q_n}$ as follows:

n	-2	-1	0	1	2	3	4	5	6
a_n			3	7	15	1	292	1	1
p_n	0	1	3	22	333	355	$103,993$	...	...
q_n	1	0	1	7	106	113	$33,102$	...	...
C_n			3	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103,993}{33,102}$	...	...

Comment. For $n \geqslant 1$, each approximation $x \approx \frac{p_n}{q_n}$ is best possible in the sense that it is better than any other approximation $\frac{a}{b}$ with $b \leqslant q_n$. In other words, if $\left|x - \frac{a}{b}\right| < \left|x - \frac{p_n}{q_n}\right|$, then $b > q_n$.

Comment. Because of this, it is natural to expect that the approximations $\frac{22}{7}$ and $\frac{355}{113}$ are particularly good, because they are followed by much "bigger" fractions.

Indeed, $\frac{22}{7} = \boxed{3.14}28...$ and $\frac{355}{113} = \boxed{3.141592}92...$ are very good approximations to π.

Comment. It is known that π is irrational, so that the above "wild" continued fraction will go on forever.

Embarrassingly, we do not know whether, for instance, $e + \pi = 5.85987448205...$ is irrational.

$e + \pi = [5; 1, 6, 7, 3, 21, 2, 1, 2, 2, 1, 1, 2, 3, 3, 2, 5, 2, 1, 1, ...]$

All evidence points to it being irrational, but nobody has a proof. (In particular, we cannot be sure that this continued fraction goes on forever.)