**Example 78. (ISBN)** The International Standard Book Number ISBN-10 consists of nine digits $a_1 a_2 \ldots a_9$ followed by a tenth check digit $a_{10}$ (the symbol $X$ is used if the digit equals $10$), which satisfies

$$a_{10} \equiv \sum_{k=1}^{9} k a_k \pmod{11}.$$

The ISBN 006085396-? is missing the check digit (printed as "?"). Compute it!

**Solution.** $1 \cdot 0 + 2 \cdot 0 + 3 \cdot 6 + 4 \cdot 0 + 5 \cdot 8 + 6 \cdot 5 + 7 \cdot 3 + 8 \cdot 9 + 9 \cdot 6 = 88 + 21 + 72 + 54 \equiv 4 \pmod{11}$
Hence, the full ISBN is 0060853964.

## 5.1 Representations of integers in different bases

**Example 79.** $25 = \boxed{1} \cdot 2^4 + \boxed{1} \cdot 2^3 + \boxed{0} \cdot 2^2 + \boxed{0} \cdot 2^1 + \boxed{1} \cdot 2^0$. We write $25 = (11001)_2$.

**Example 80.** We are commonly using the **decimal system** of writing numbers:

$$1234 = 4 \cdot 10^0 + 3 \cdot 10^1 + 2 \cdot 10^2 + 1 \cdot 10^3.$$

$10$ is called the base, and $1, 2, 3, 4$ are the digits in base $10$. To emphasize that we are using base $10$, we will write $1234 = (1234)_{10}$. Likewise, we write

$$(1234)_b = 4 \cdot b^0 + 3 \cdot b^1 + 2 \cdot b^2 + 1 \cdot b^3.$$

In this example, $b > 4$, because, if $b$ is the base, then the digits have to be in $\{0, 1, \ldots, b - 1\}$.

**Example 81.** Express $25$ in base $2$.

**Solution.** We already noticed that $25 = 16 + 8 + 1 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$. Hence, $25 = (11001)_2$.
Alternatively, here's how we could have determined the digits without prior knowledge:

- $25 = 12 \cdot 2 + \boxed{1}$. Hence, $25 = (\ldots 1)_2$ where $\ldots$ are the digits for $12$.
- $12 = 6 \cdot 2 + \boxed{0}$. Hence, $25 = (\ldots 01)_2$ where $\ldots$ are the digits for $6$.
- $6 = 3 \cdot 2 + \boxed{0}$. Hence, $25 = (\ldots 001)_2$ where $\ldots$ are the digits for $3$.
- $3 = 1 \cdot 2 + \boxed{1}$, with $\boxed{1}$ left over. Hence, $25 = (11001)_2$.

**Example 82.** Express $49$ in base $2$.

**Solution.**

- $49 = 24 \cdot 2 + \boxed{1}$. Hence, $49 = (\ldots 1)_2$ where $\ldots$ are the digits for $24$.
- $24 = 12 \cdot 2 + \boxed{0}$. Hence, $49 = (\ldots 01)_2$ where $\ldots$ are the digits for $12$.
- $12 = 6 \cdot 2 + \boxed{0}$. Hence, $49 = (\ldots 001)_2$ where $\ldots$ are the digits for $6$.
- $6 = 3 \cdot 2 + \boxed{0}$. Hence, $49 = (\ldots 0001)_2$ where $\ldots$ are the digits for $3$.
- $3 = 1 \cdot 2 + \boxed{1}$, with $\boxed{1}$ left over. Hence, $49 = (110001)_2$.

**Other bases.** What is $49$ in base $3$? $49 = 16 \cdot 3 + \boxed{1}$, $16 = 5 \cdot 3 + \boxed{1}$, $5 = 1 \cdot 3 + \boxed{2}$, $\boxed{1}$. Hence, $49 = (1211)_3$.
What is $49$ in base $7$? $49 = (10)_7$.


Armin Straub
straub@southalabama.edu

**Example 83.** Bases $2$, $8$ and $16$ (binary, octal and hexadecimal) are commonly used in computer applications.

For instance, in JavaScript or Python, $0\text{b}...$ means $(...)_2$, $0\text{o}...$ means $(...)_8$, and $0\text{x}...$ means $(...)_{16}$.

The digits $0, 1, ..., 15$ in hexadecimal are typically written as $0, 1, ..., 9, A, B, C, D, E, F$.

**Problem.** Which number is $0\text{xD1}$?

**Solution.** $0\text{xD1} = 13 \cdot 16 + 1 = 209$.

The South Alabama Jaguar NCAA team color code is $0\text{xD11241}$. That means $\text{RGB}(209, 18, 65)$, where each value (ranging from $0$ to $255$) quantifies the amount of red (R), green (G) and blue (B).

For instance, $0\text{x000000}$ is black, and $0\text{xFF0000}$ is red, and $0\text{xFFFFFF}$ is white.

We can thus see that the color $0\text{xD11241}$ is close to a red (though not a pure one).

**Example 84.** **(divisibility by $9$)** A number $n = (a_m a_{m-1} \cdots a_0)_{10}$ is divisible by $9$ if and only if the sum of its decimal digits $a_m + a_{m-1} + \ldots + a_0$ is divisible by $9$.

**Why?** Note that $10^r \equiv 1^r \equiv 1 \pmod{9}$ for any integer $r \geqslant 0$.

In particular, $n = a_m \cdot 10^m + a_{m-1} \cdot 10^{m-1} + \ldots + a_1 \cdot 10^1 + a_0 \equiv a_m + a_{m-1} + \ldots + a_1 + a_0 \pmod{9}$.

**For instance.** $1234567$ is not divisible by $9$ because $1 + 2 + 3 + \ldots + 7 = \frac{7(7+1)}{2} = 28$ is not divisible by $9$. In fact, $1234567 \equiv 28 \equiv 1 \pmod{9}$.

**Example 85.** **(divisibility by $11$)** A number $n = (a_m a_{m-1} \cdots a_0)_{10}$ is divisible by $11$ if and only if the alternating sum of its decimal digits $(-1)^m a_m + (-1)^{m-1} a_{m-1} + \ldots + a_0$ is divisible by $11$.

**Why?** Note that $10^r \equiv (-1)^r \pmod{11}$ for any integer $r \geqslant 0$. In particular,

$n = a_m \cdot 10^m + a_{m-1} \cdot 10^{m-1} + \ldots + a_1 \cdot 10^1 + a_0 \equiv (-1)^m a_m + (-1)^{m-1} a_{m-1} + \ldots - a_1 + a_0 \pmod{11}$.

**For instance.** $123456$ is not divisible by $11$ because $6 - 5 + 4 - 3 + 2 - 1 = 3$ is not divisible by $11$. In fact, $123456 \equiv 3 \pmod{11}$.

**Example 86.** Using binary exponentiation, compute $5^{49} \pmod{105}$.

**Solution.** Recall that $49 = (110001)_2 = 2^0 + 2^4 + 2^5$.

$5^1 = 5$, $5^2 = 25$, $5^4 = 25^2 = 625 \equiv -5$, $5^8 \equiv (-5)^2 = 25$, $5^{16} \equiv 25^2 \equiv -5$, $5^{32} \equiv (-5)^2 = 25$

Hence, $5^{49} = 5^{32} \cdot 5^{16} \cdot 5^1 \equiv 25 \cdot (-5) \cdot 5 \equiv 5$.

**Alternative solution.** If we prefer, we can be a tiny bit more efficient by computing the binary expansion and powers at once:

$5^{49} = 5 \cdot 5^{2 \cdot 24} = 5 \cdot 25^{24} = 5 \cdot (25^2)^{12} \equiv 5 \cdot (-5)^{12} = 5 \cdot 25^6 = 5 \cdot (25^2)^3 \equiv 5 \cdot (-5)^3 \equiv -5$

Armin Straub
straub@southalabama.edu