Example 15. Let us prove that $F_n < 2^n$ for all integers $n \geqslant 0$.

Getting a feeling. $0 < 1$, $1 < 2$, $1 < 4$, $2 < 8$, $3 < 16$, $5 < 32$, $8 < 64$ (seems like the claim is "very" true)

However, the "however" remark on Fibonacci numbers from last time implies that $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi \approx 1.618$.
In other words, F_n is indeed growing exponentially (but $1.618 < 2$)!
(In particular, say, $F_n > n^{1000}$ for large enough n, so we should be careful only looking at the first few cases.)

Proof.

- base cases: $F_0 = 0 < 2^0 = 1$, $F_1 = 1 < 2^1 = 2$.
- induction step: suppose that $F_m < 2^m$ for all integers $m \in \{1, 2, \ldots, n\}$. (strong induction!)
  We need to show that $F_{n+1} < 2^{n+1}$.
  $F_{n+1} = F_n + F_{n-1} \overset{(\mathrm{IH})}{<} 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}$ □

Important note. Why was it necessary to consider two base cases?

1.4 The binomial theorem

$n!$ counts the number of ways n objects can be ordered.
The **binomial coefficient**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

counts the number of ways in which we can select k elements from a total of n elements.

Example 16. $\binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$

Theorem 17. (Pascal's rule) For integers n, k, such that $n \geqslant 0$ and $k \geqslant 1$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Proof. Let us divide both sides of the claimed identity by $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$, and write everything in terms of factorials:

$$\frac{(n+1)!}{k!(n-k+1)!} \frac{(k-1)!(n-k+1)!}{n!} \overset{?}{=} \frac{n!}{k!(n-k)!} \frac{(k-1)!(n-k+1)!}{n!} + 1$$

(The $\overset{?}{=}$ reminds us that we are working towards proving this identity.) Cancelling terms, this is equivalent to

$$\frac{n+1}{k} \overset{?}{=} \frac{n-k+1}{k} + 1.$$

This latter equation is obviously true. □

Example 18. This gives rise to **Pascal's triangle**

$$\begin{array}{ccccccccc} & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ & & & & \ldots & & & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \ldots & & & & \end{array}$$

Note that each element is the sum of the two elements above it (that's what Pascal's rule is saying).

Example 19. Let us expand $(x+y)^n$.

$$\begin{aligned}(x+y)^1 &= x+y\\(x+y)^2 &= x^2+2xy+y^2\\(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

The coefficients are exactly the numbers from Pascal's triangle!

Of course, that's just a conjecture at this point. But we will prove it below.

Theorem 20. (Binomial theorem) For any integer $n\geqslant 1$,

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

Proof. (by induction) We prove the claim by induction on n.

- **(base case)** $(x+y)^1=\binom{1}{0}x+\binom{1}{1}y$ verifies that the claim is true for $n=1$.
- **(induction step)** Assume that $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$ is true for some n.

  We need to show that $(x+y)^{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}x^ky^{n+1-k}$.

$$\begin{aligned}(x+y)^{n+1} &= (x+y)(x+y)^n\\\text{(using the induction hypothesis)} &= (x+y)\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}\\&= \underbrace{\sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k}}_{=\sum_{k=1}^{n+1}\binom{n}{k-1}x^ky^{n+1-k}}+\sum_{k=0}^{n}\binom{n}{k}x^ky^{n+1-k}\\&= x^{n+1}+y^{n+1}+\sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]x^ky^{n+1-k}\\\text{(Pascal's rule)} &= x^{n+1}+y^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}x^ky^{n+1-k}\\&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^ky^{n+1-k}\end{aligned}$$

  That's what we had to prove! □

Proof. (combinatorial) This alternative proof assumes that we know that $\binom{n}{k}$ counts the number of ways in which we can select k elements from a total of n elements.

[Here is one way to see this from the definition $\binom{n}{k}=\frac{n!}{k!(n-k)!}$. We wish to count the number of ways in which we can select k elements from a total of n elements. There are $n!$ ways to line up the n elements in order. Our intention is to select the first k elements. However, different ways to order the n elements will result in the same selection. Namely, the order of the first k doesn't matter ($k!$ such orderings), and the order of the remaining $n-k$ does not matter ($(n-k)!$ such orderings).]

Note that all of the terms we get when expanding $(x+y)^n=(x+y)(x+y)\cdots(x+y)$ will be of the form x^ky^{n-k} for some $k\in\{0,1,\ldots,n\}$. So, how often will the term x^ky^{n-k} come up? For each factor $x+y$, we need to decide whether to choose x or y. We get x^ky^{n-k} in the end, if we choose x in exactly k of the n factors. There is $\binom{n}{k}$ many such possibilities. □

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