Example 11. We are interested in the sums $s(n) = 1 + 2 + 4 + \ldots + 2^n$.

Getting a feeling. $s(1) = 1 + 2 = 3$, $s(2) = 1 + 2 + 4 = 7$, $s(3) = 1 + 2 + 4 + 8 = 15$, $s(4) = 31$

Conjecture. $s(n) = 2^{n+1} - 1$.

Proof by induction. The statement we want to prove by induction is: $s(n) = 2^{n+1} - 1$ for all integers $n \geqslant 1$.

- **(base case)** $s(1) = 1 = 2^{1+1} - 1$ verifies that the claim is true for $n = 1$.
- **(induction step)** Assume that $s(n) = 2^{n+1} - 1$ is true for some n.
  We need to show that $s(n+1) = 2^{n+2} - 1$.
  Using the induction hypothesis, $s(n+1) = s(n) + 2^{n+1} \overset{\text{IH}}{=} (2^{n+1} - 1) + 2^{n+1} = 2^{n+2} - 1$. QED!

Direct proof. $2s(n) = 2(1 + 2 + 4 + \ldots + 2^n) = 2 + 4 + \ldots + 2^{n+1} = s(n) - 1 + 2^{n+1}$. Hence, $s(n) = 2^{n+1} - 1$.

Example 12. Can we generalize the previous example by replacing 2 with x?
That is, we are now interested in the sums $s(n) = 1 + x + x^2 + \ldots + x^n$.

Mimic previous direct approach. $x s(n) = x(1 + x + x^2 + \ldots + x^n) = x + x^2 + \ldots + x^{n+1} = s(n) - 1 + x^{n+1}$.
Hence, $(x-1)s(n) = x^{n+1} - 1$, and we have found:

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \qquad \textbf{(geometric sum)}$$

Sigma notation. Instead of $1 + x + x^2 + \ldots + x^n$ we will begin to write $\sum_{k=0}^{n} x^k$.

Geometric series. We can let $n \to \infty$ to get $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, provided that $|x| < 1$.

Example 13. **(Homework)** Prove the formula for geometric sums using induction.

(mathematical induction, strong form) To prove that $\mathrm{CLAIM}(n)$ is true for all integers $n \geqslant n_0$, it suffices to show:

- **(base case)** $\mathrm{CLAIM}(n_0)$ is true.
- **(induction step)** if $\mathrm{CLAIM}(m)$ is true for $m \in \{n_0, n_0 + 1, \ldots, n\}$ for some n, then $\mathrm{CLAIM}(n+1)$ is true as well.

Example 14. Induction is not only a proof technique but also a common way to define things.

- The **factorial** $n!$ can be defined inductively (i.e. recursively) by

$$0! = 1, \quad (n+1)! = n! \cdot (n+1).$$

  Comment. This may not seem impressive, because we can "spell out" $n! = 1 \cdot 2 \cdot 3 \cdots (n-1)n$ directly.

- The **Fibonacci numbers** F_n are defined inductively (i.e. recursively) by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}.$$

  Getting a feeling. $F_2 = F_1 + F_0 = 1$, $F_3 = F_2 + F_1 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, ...

  Comment. In contrast to the factorial, there is no immediate way to "spell out" F_n directly. In other words, to compute F_{20}, you would first compute F_{19} and F_{18} (and then F_{17}, F_{16} and so on).

  However. Though not at all obvious, there is a way to compute F_n directly. Let $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$. Then $F_n = \lfloor \varphi^n / \sqrt{5} \rceil$. Try it! For instance, $\varphi^{10} / \sqrt{5} \approx 55.0036$. That seems like magic at first. But it is the beginning of a general theory (look up, for instance, Binet's formula and C-finite sequences).

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