Homework #3

MATH 311 — Intro to Number Theory
due in class on Thursday, Sep 15

Please print your name:

These problems are not suited to be done last minute!
Also, if you start early, you can consult with me if you should get stuck.

Problem 1.

(a) Find $d = \gcd(100, 2016)$. Using the Euclidean algorithm, find integers x, y such that $100x + 2016y = d$.

(b) Find $d = \gcd(100, 2017)$. Using the Euclidean algorithm, find integers x, y such that $100x + 2017y = d$.

Solution.

(a) $\underbrace{\gcd(100, 2016)}_{2016 = 20 \cdot 100 + 16} = \underbrace{\gcd(16, 100)}_{100 = 6 \cdot 16 + 4} = \gcd(4, 16) = 4.$

Tracing back through the algorithm, we find

$4 = \underbrace{100 - 6 \cdot 16}_{16 = 2016 - 20 \cdot 100} = 121 \cdot 100 - 6 \cdot 2016.$ That is, $x = 121$ and $y = -6$ work.

[Note that the general solution to $100x + 2016y = 4$ therefore is $x = 121 + 2016t$ and $y = -6 - 100t$.]

(b) $\underbrace{\gcd(100, 2017)}_{2017 = 20 \cdot 100 + 17} = \underbrace{\gcd(17, 100)}_{100 = 5 \cdot 17 + 15} = \underbrace{\gcd(15, 17)}_{17 = 1 \cdot 15 + 2} = \underbrace{\gcd(2, 15)}_{15 = 7 \cdot 2 + 1} = \gcd(1, 2) = 1.$

Tracing back through the algorithm, we find

$1 = \underbrace{15 - 7 \cdot 2}_{2 = 17 - 1 \cdot 15} = \underbrace{8 \cdot 15 - 7 \cdot 17}_{15 = 100 - 5 \cdot 17} = \underbrace{8 \cdot 100 - 47 \cdot 17}_{17 = 2017 - 20 \cdot 100} = 948 \cdot 100 - 47 \cdot 2017.$ That is, $x = 948$ and $y = -47$ work.

[Note that the general solution to $100x + 2017y = 1$ therefore is $x = 948 + 2017t$ and $y = -47 - 100t$.] □

Problem 2.

(a) For which values of k has the diophantine equation $24x + 138y = k$ at least one integer solution?

(b) Determine all integer solutions of $24x + 138y = 18$.

Solution.

(a) $\gcd(24, 138) = \gcd(18, 24) = \gcd(6, 18) = 6$

Hence, the diophantine equation $24x + 138y = k$ has solutions if and only if $6|k$.

(b) Since $6|18$ there will be solutions. We first simplify the equation to $4x + 23y = 3$.

As a first step, we find a particular solution to $4x + 23y = 1$ (this is possible since, as a consequence of dividing the equation by the gcd, we have $\gcd(4, 23) = 1$) using the Euclidean algorithm:

$$\underbrace{\gcd(4, 23)}_{23 = 5 \cdot 4 + 3} = \underbrace{\gcd(3, 4)}_{4 = 1 \cdot 3 + 1} = \gcd(1, 3) = 1.$$

We trace back through the algorithm to find

$$1 = \underbrace{4 - 1 \cdot 3}_{3 = 23 - 5 \cdot 4} = -1 \cdot 23 + 6 \cdot 4.$$

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In other words, $4x + 23y = 1$ has the solution $x = 6$, $y = -1$.

Consequently, a particular solution to $4x + 23y = 3$ is $x = 3 \cdot 6 = 18$, $y = 3 \cdot (-1) = -3$.

Finally, the general solution to $4x + 23y = 3$ is $x = 18 + 23t$, $y = -3 - 4t$ where the free parameter t can be any integer. □

Problem 3. The neighborhood theater charges \$1.80 for adult admissions and \$.75 for children. On a particular evening the total receipts were \$90. Assuming that more adults than children were present, how many people attended?

Solution. Let x be the number of adults, and y the number of children. On that particular night,

$$1.8x + 0.75y = 90, \qquad \text{or, equivalently,} \quad 180x + 75y = 9000.$$

Since $\gcd(180, 75) = \gcd(30, 75) = \gcd(15, 30) = 15$ and $15|9000$, this diophantine equation will have integer solutions. We first simplify it, by dividing everything by 15, to get $12x + 5y = 600$.

As a first step, we find a particular solution to $12x + 5y = 1$ (this is possible since, as a consequence of dividing the equation by the gcd, we have $\gcd(12, 5) = 1$) using the Euclidean algorithm:

$$\underbrace{\gcd(12, 5)}_{12 = 2\cdot 5 + 2} = \underbrace{\gcd(2, 5)}_{5 = 2\cdot 2 + 1} = \gcd(1, 2) = 1.$$

We trace back through the algorithm to find

$$1 = \underbrace{5 - 2 \cdot 2}_{2 = 12 - 2\cdot 5} = -2 \cdot 12 + 5 \cdot 5.$$

In other words, $12x + 5y = 1$ has the solution $x = -2$, $y = 5$.

Consequently, a particular solution to $12x + 5y = 600$ is $x = 600 \cdot (-2) = -1200$, $y = 600 \cdot 5 = 3000$.

Therefore, the general solution to $12x + 5y = 600$ is $x = -1200 + 5t$, $y = 3000 - 12t$ where the free parameter t can be any integer.

The assumption that more adults than children were present translates into $x > y$, and we also have $y \geqslant 0$ because the number of children cannot be negative.

$y \geqslant 0$ means $12t \leqslant 3000$, that is, $t \leqslant 250$.

$x > y$ means that $-1200 + 5t > 3000 - 12t$, that is, $17t > 4200$ or $t > 247 + \frac{1}{17}$.

This leaves the possibilities $t \in \{248, 249, 250\}$.

- $t = 248$ corresponds to $x = 40$ and $y = 24$.
- $t = 249$ corresponds to $x = 45$ and $y = 12$.
- $t = 250$ corresponds to $x = 50$ and $y = 0$.

Hence, the number of people attending was either 64, 57 or 50. □

Problem 4.

(a) Show that $(2, 3)$ is the only pair (p_1, p_2) of primes such that $p_2 = p_1 + 1$.

(b) A pair of primes (p_1, p_2) is a twin prime pair if $p_2 = p_1 + 2$. Show that every twin prime pair except $(3, 5)$ is of the form $(6n - 1, 6n + 1)$.

[*Hint:* Write the pair as $(N - 1, N + 1)$ and think about the possible remainders of N upon division by 6.]

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(c) Show that $(2,5)$ is the only pair (p_1, p_2) of primes such that $p_2 = p_1 + 3$.

(d) Write down a few pairs (p_1, p_2) of primes such that $p_2 = p_1 + 4$.

Solution.

(a) Either p_1 or $p_2 = p_1 + 1$ is an even number. Since the only even prime is 2 and because $(1,2)$ is not a pair of primes, the pair $(2,3)$ is the only pair (p_1, p_2) of primes such that $p_2 = p_1 + 1$.

(b) The question is to show that all pairs of primes of the form $(N - 1, N + 1)$ can be written in the form $(6n-1, 6n+1)$. In other words, we need to show that $6|N$. Since we are talking about divisibility by 6, it is natural to consider the possible remainders of N upon division by 6:

- $N = 6q$: clearly, $6|N$ which is what we claim happens always (with one exception). So there is nothing to show in this case.

- $N = 6q+1$: then the pair is $(6q, 6q+2)$, but this can never be a prime pair (no prime is divisible by 6).

- $N = 6q+2$: then the pair is $(6q+1, 6q+3)$, which is never a prime pair ($3|6q+3$, so $6q+3$ is a prime only if $q = 0$, but $(1,3)$ is not a pair of primes).

- $N = 6q+3$: then the pair is $(6q+2, 6q+4)$, which is never a prime pair (these are two even numbers and there is only one even prime).

- $N = 6q+4$: then the pair is $(6q+3, 6q+5)$. Since $3|6q+3$, the number $6q+3$ is a prime only if $q = 0$. Indeed, for $q = 0$, we obtain the single prime pair $(3,5)$.

- $N = 6q+5$: then the pair is $(6q+4, 6q+6)$, which is never a prime pair (these are two even numbers and there is only one even prime).

(c) Either p_1 or $p_2 = p_1 + 3$ is an even number. Since the only even prime is 2, the pair $(2,5)$ is the only pair (p_1, p_2) of primes such that $p_2 = p_1 + 3$.

(d) $(3,7)$, $(7,11)$, $(13,17)$, $(19,23)$, $(37,41)$, ...

It is conjectured that there should be infinitely many such pairs, but nobody is currently able to prove that. ☐

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