Homework #1

MATH 311 — Intro to Number Theory
due in class on Thursday, Sep 1

Please print your name:

These problems are not suited to be done last minute!
Also, if you start early, you can consult with me if you should get stuck.

Problem 1. Using induction, prove that $1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Solution. Write $s(n) = 1^3 + 2^3 + \ldots + n^3$.
We use induction on the claim that $s(n) = \left(\frac{n(n+1)}{2}\right)^2$.

- The base case $(n = 1)$ is that $s(1) = 1^3 = \left(\frac{1(2)}{2}\right)^2$. That's true.
- For the inductive step, assume the formula holds for some value of n.

  We need to show the formula also holds for $n + 1$.

$$\begin{aligned} s(n+1) &= s(n) + (n+1)^3 \\ \text{(using the induction hypothesis)} &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\ &= \frac{(n+1)^2}{4}[n^2 + 4(n+1)] \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2 \end{aligned}$$

  This shows that the formula also holds for $n + 1$.

By induction, the formula is true for all integers $n \geqslant 1$. □

Problem 2.

(a) Experiment to find a formula for $1 + 3 + 5 + \ldots + (2n + 1)$.

(b) Prove that formula using induction.

(c) Can you give a second (direct) proof using Gauss' formula for $1 + 2 + 3 + \ldots + n$?

Solution. Write $s(n) = 1 + 3 + 5 + \ldots + (2n + 1)$.

(a) $s(0) = 1$, $s(1) = 1 + 3 = 4$, $s(2) = 1 + 3 + 5 = 9$, $s(3) = 16$, $s(4) = 25$

It seems that $s(n) = (n + 1)^2$.

(b) We use induction on the claim that $s(n) = (n + 1)^2$.

- We have already check that the base case for $n = 1$ is true.
- For the inductive step, assume the formula holds for some value of n.

Armin Straub
straub@southalabama.edu

We need to show the formula also holds for $n+1$.

$$\begin{aligned} s(n+1) &= s(n)+2(n+1)+1 \\ \text{(using the induction hypothesis)} &= (n+1)^2+2n+3 \\ &= n^2+4n+4 \\ &= (n+2)^2 \end{aligned}$$

This shows that the formula also holds for $n+1$.

By induction, the formula is true for all integers $n \geqslant 1$.

(c) $s(n) = \sum_{k=0}^{n} (2k+1) = \sum_{k=0}^{n} (2k) + \sum_{k=0}^{n} 1 = 2\sum_{k=0}^{n} k + (n+1) \overset{\text{(Gauss)}}{=} 2\frac{n(n+1)}{2} + (n+1) = (n+1)^2$

Alternative. We can also proceed like little Gauss:

$$\begin{aligned} 2s(n) &= 1+3+5+\ldots+(2n+1) \\ &+ (2n+1)+\ldots+5+3+1 \\ &= \underbrace{(2n+2)+(2n+2)+\ldots+(2n+2)}_{n+1 \text{ many terms}} \\ &= (n+1)(2n+2) = 2(n+1)^2 \end{aligned}$$

Dividing by 2, we obtain $s(n) = (n+1)^2$.

□

Problem 3. Consider the sequence a_n defined recursively by $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, and

$$a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3}.$$

(a) Compute a_7. [Your answer should be 169.]

(b) Come up with a conjecture that relates the numbers a_n with a sequence we have already seen in class.
If you get stuck, just ask me! I will be happy to give hints as needed.

(c) **(nothing to do here)** Unfortunately, it is rather tricky to prove your conjecture. However, you are certainly invited to try (and get some bonus).

Instead, we aim for lower hanging fruit: (Both claims that follow are clearly true if we could prove our conjecture.)

(d) Using (strong) induction, prove that the sequence a_n is increasing (that is, $a_n \geqslant a_{n-1}$ for all integers $n \geqslant 1$).

(e) Using (strong) induction, prove that $a_n < 4^n$ for all integers $n \geqslant 0$.
Note that, by the previous part, we now know $a_n \geqslant 0$. (Why is this **not** completely obvious from the definition?)

Caution. For the last two parts, how many base cases need to be considered?

Solution.

(a) $a_3 = 2a_2 + 2a_1 - a_0 = 4$

$a_4 = 2a_3 + 2a_2 - a_1 = 9$

$a_5 = 2a_4 + 2a_3 - a_2 = 25$

$a_6 = 2a_5 + 2a_4 - a_3 = 64$

Armin Straub
straub@southalabama.edu

$a_7 = 2a_6 + 2a_5 - a_4 = 169$

(b) We note that all values appear to be squares: $0^2, 1^2, 1^2, 2^2, 3^2, 5^2, 8^2, 13^2, \ldots$

And the square roots appear to be the Fibonacci numbers F_n!

Conjecture. For all integers $n \geqslant 0$, we have $a_n = F_n^2$.

(c) **(base case)** We have already checked the conjecture for $n \in \{0, 1, \ldots, 7\}$.

(induction step) Assume that $a_m = F_m^2$ for all $m \in \{0, 1, \ldots, n\}$. We need to show that $a_{n+1} = F_{n+1}^2$.

$$\begin{aligned} a_{n+1} &= 2a_n + 2a_{n-1} - a_{n-2} \quad \text{(by definition)} \\ &= 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \quad \text{(by the induction hypothesis)} \\ &= 2(F_n + F_{n-1})^2 - 4F_nF_{n-1} - F_{n-2}^2 \\ &= 2F_{n+1}^2 - 4F_nF_{n-1} - F_{n-2}^2 \\ &= 2F_{n+1}^2 - 4(F_{n-1} + F_{n-2})F_{n-1} - F_{n-2}^2 \\ &= 2F_{n+1}^2 - \underbrace{(2F_{n-1} + F_{n-2})}_{= F_{n-1} + F_n = F_{n+1}}{}^2 \\ &= F_{n+1}^2 \end{aligned}$$

By induction, $a_n = F_n^2$ for all integers $n \geqslant 1$.

(d) **(base case)** $a_1 = 1 \geqslant a_0 = 0$, $a_2 = 1 \geqslant a_1 = 1$ and $a_3 = 4 \geqslant a_2 = 1$.

(induction step) Assume that $a_m \geqslant a_{m-1}$ for all $m \in \{0, 1, \ldots, n\}$. We need to show that $a_{n+1} \geqslant a_n$.

$$\begin{aligned} a_{n+1} &= 2a_n + 2a_{n-1} - a_{n-2} \quad \text{(by definition)} \\ &= 2a_n + a_{n-1} + \underbrace{(a_{n-1} - a_{n-2})}_{\geqslant 0 \text{ (by IH)}} \\ &\geqslant 2a_n + a_{n-1} \end{aligned}$$

A weak consequence of the induction hypothesis is that $a_n \geqslant 0$ and $a_{n-1} \geqslant 0$. Hence,

$$a_{n+1} \geqslant 2a_n + a_{n-1} = a_n + \underbrace{(a_n + a_{n-1})}_{\geqslant 0} \geqslant a_n,$$

which is what we needed to show.

By induction, $a_n \geqslant a_{n-1}$ for all integers $n \geqslant 1$.

(e) **(base case)** $a_0 = 0 < 4^0$, $a_1 = 1 < 4^1$ and $a_2 = 1 < 4^2$.

(induction step) Assume that $a_m < 4^m$ for all $m \in \{0, 1, \ldots, n\}$. We need to show that $a_{n+1} < 4^{n+1}$.

$$\begin{aligned} a_{n+1} &= 2a_n + 2a_{n-1} - \underbrace{a_{n-2}}_{\geqslant 0} \quad \text{(by definition)} \\ &\leqslant 2a_n + 2a_{n-1} \\ &< 2 \cdot 4^n + 2 \cdot 4^{n-1} \quad \text{(by the induction hypothesis)} \\ &< 2 \cdot 4^n + 2 \cdot 4^n \\ &= 4^{n+1}, \end{aligned}$$

which is what we needed to show.

By induction, $a_n < 4^n$ for all integers $n \geqslant 0$. □

Armin Straub
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