

Review. Complex numbers

Example 155. (April Fools' Day!) Foul play with complex numbers:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = ii = -1.$$

When using the principal square-root (which basically takes the positive root, that is, the one with positive real part), the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ does not hold universally (so the trouble lies with the third equality). It does hold if, for instance, $a \geq 0$ or $b \geq 0$. Apparently, this trouble historically resulted in controversy around complex numbers, with some mathematicians rejecting them outright.

Example 156. (April Fools' Day!) What is the norm of the vector $\begin{bmatrix} 1 \\ i \end{bmatrix}$? Could it be zero?!

Solution. $\left\| \begin{bmatrix} 1 \\ i \end{bmatrix} \right\| = \sqrt{|1|^2 + |i|^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

Comment. If we carelessly use $\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$ (which works fine only when a and b are real numbers), then we would (incorrectly) get that the vector has norm 0.

Example 157. What is the norm of the vector $\begin{bmatrix} 4-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 4-i \\ 2+3i \end{bmatrix} \right\| = \sqrt{|4-i|^2 + |2+3i|^2} = \sqrt{(4^2 + (-1)^2) + (2^2 + 3^2)} = \sqrt{30}$

Example 158. True or false? Every $n \times n$ matrix A always has an eigenvector v .

Solution. True! Every $n \times n$ matrix always has exactly n eigenvalues (if we allow complex eigenvalues and count with repetition).

If λ is one of those eigenvalues, then the dimension of the λ -eigenspace is at least 1 (because $\det(A - \lambda I) = 0$ so that $Av = \lambda v$ has nonzero solutions v).

Example 159. (April Fools' Day!) Let A be an $n \times n$ matrix. By the previous example, we can find an eigenvector v and an eigenvalue λ such that $Av = \lambda v$.

Let us rewrite that as $Av = \lambda Iv$ where I is the $n \times n$ identity matrix. Does it follow that $A = \lambda I$?

Solution. No, that is (of course) not a valid conclusion. One cannot cancel a vector v in this way.

Recall, more generally for matrices, that in order to conclude from $AB = CB$ that $A = C$ we need extra information on B such as B being invertible.