Another brief look at projections (and reflections)

(projections) Suppose that M is the projection matrix for projecting onto a subspace W.

- The 1-eigenspace of M is W.
- The 0-eigenspace of M is $W^\perp$.

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get projected to themselves. But those are precisely the vectors in W (recall that projecting a vector $\boldsymbol{v}$ onto W means producing the vector in W that is closest to $\boldsymbol{v}$). Can you likewise spell out the situation for the 0-eigenspace?

Note that M cannot have further eigenvalues (because the dimensions of W and $W^\perp$ already add up to the dimension of the space that we are working in).

Because the eigenvalues of M are real and the eigenspaces are orthogonal, the matrix M has a diagonalization of the form $M = PDP^T$ (make sure you can explain why!) which implies that M is symmetric (that's because $M^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = M$).

Example 117. Let A be the matrix for orthogonally projecting onto $W = \text{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^{-1}$.

(b) Diagonalize A as $A = PDP^T$.

Comment. This gives us yet another way to get our hands on projection matrices: we can directly write down the matrices P, D for the diagonalization $A = PDP^T$. The main point here is that the diagonalization of a A nicely reveals all the information about the projection.

[Can you see that this is not really a "new" way of computing projection matrices? In particular, note that, if Q is the matrix P with the third column omitted, then $A = QQ^T$.]

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional).

We already have a basis for W. On the other hand, $W^\perp = \text{null}\left(\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4 & 0 & -1/4 \\ 0 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (of the 1-eigenspace), we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{17} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{17} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$.

Next, we normalize the vectors $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{17}\begin{bmatrix} -4 \\ 34 \\ 16 \end{bmatrix}, \begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$ to $\frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{357}} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}, \frac{1}{\sqrt{21}} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4/\sqrt{17} & -2/\sqrt{357} & -1/\sqrt{21} \\ 0 & 17/\sqrt{357} & -2/\sqrt{21} \\ 1/\sqrt{17} & 8/\sqrt{357} & 4/\sqrt{21} \end{bmatrix}$.

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By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{21}\begin{bmatrix} 20 & -2 & 4 \\ -2 & 17 & 8 \\ 4 & 8 & 5 \end{bmatrix}$ as in Example 55.

Example 118. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional). Note that we are lucky and already have an orthogonal basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is not invertible (because 0 is an eigenvalue) and therefore also cannot be orthogonal. Like any projection matrix, A is symmetric.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

(reflections) Suppose that M is the matrix for reflecting through the plane W in 3-space.

- The 1-eigenspace of M is W. (dimension 2)
- The -1-eigenspace of M is $W^\perp$. (dimension 1)

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get reflected to themselves. But those are precisely the vectors in the plane W (only vectors on the plane are unchanged by the reflection). On the other hand, the -1-eigenspace consists of those vectors $\boldsymbol{v}$ that get reflected to $-\boldsymbol{v}$ (the exact opposite direction). These are precisely the vectors orthogonal to the plane.

As in the case of projection matrices, because the eigenvalues are real and the eigenspaces are orthogonal, the reflection matrices are symmetric.

Comment. In this context, the line $W^\perp$ is often called the **normal line** of the plane W.

Example 119. Let A be the matrix for reflecting through the plane $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

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Solution.

(a) The eigenvalues of A are $1,1,-1$. The 1-eigenspace of A is W, and the -1-eigenspace is $W^\perp$.

In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

As in the previous example, $W^\perp = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is invertible (because 0 is not an eigenvalue).

Like any reflection matrix, A is symmetric.

Finally, note that $A^2 = I$ (reflecting twice isn't doing anything), so that $A^{-1} = A$. It follows that A is orthogonal, because $A^{-1} = A = A^T$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. Similarly, a $n \times n$ matrix corresponds to a reflection (through a hyperplane) if and only if it has a $(n-1)$-dimensional 1-eigenspace and a 1-dimensional -1-eigenspace and these two spaces are orthogonal.

An alternative way of computing reflection matrices. Realize that, if $\boldsymbol{n}$ is the vector orthogonal to the plane (i.e. $\boldsymbol{n}$ is the normal vector of the plane), then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

We already observed that $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hence, the reflection of $\boldsymbol{v}$ is $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n}) = \boldsymbol{v} - 2\boldsymbol{n}\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{\boldsymbol{n}\cdot\boldsymbol{n}} = \boldsymbol{v} - 2\frac{\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{v}}{\boldsymbol{n}^T\boldsymbol{n}} = \left(I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}}\right)\boldsymbol{v}$.

Accordingly, the reflection matrix is $A = I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. In other words, we got A from subtracting 2 times the projection matrix onto $\boldsymbol{n}$ from I.