Review. Recurrence equations, diagonalization, explicit (Binet-like) formulas

Example 116. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 5a_n$ and $a_0 = 0$, $a_1 = 1$.

(a) Determine the first few terms of the sequence.

(b) Find an explicit (Binet-like) formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, 1, 2, 9, 28, 101, 342, 1189, 4088, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{6}$.

Hence, $a_n = C_1(1+\sqrt{6})^n + C_2(1-\sqrt{6})^n$ and we only need to figure out the values of C_1 and C_2.
Using the two initial conditions, we get two equations:
$(a_0 =)\ C_1 + C_2 = 0$, $(a_1 =)\ C_1(1+\sqrt{6}) + C_2(1-\sqrt{6}) = 1$.
Solving, we find $C_1 = \frac{1}{2\sqrt{6}}$ and $C_2 = -\frac{1}{2\sqrt{6}}$ so that, in conclusion, $a_n = \frac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$.

Comment. Alternatively, we could have proceeded as we did previously in the case of the Fibonacci numbers: starting with the recursion matrix M, we compute its diagonalization $M = PDP^{-1}$. Multiplying out $PD^nP^{-1}\begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{6} \approx 3.44949$.

Comment. Actually, we don't need the Binet-like formula for this conclusion. Just the eigenvalues and the observation that C_1 cannot be 0 are enough. [We cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{6})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

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