Example 110. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -1$, $a_1 = 5$.

   (a) Determine the first few terms of the sequence.

   (b) Write down a matrix-vector version of the recursion.

   (c) Find an explicit formula for a_n.

   (d) Determine $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n}$.

Solution.

   (a) $-1, 5, 7, 29, 79, 245, 727, 2189, 6559, \ldots$

   (b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

   (c) **(solution using matrix powers)** Thus, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.

After some work (do it!), we diagonalize $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$ with $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix}}_{= \begin{bmatrix} 3^{n+1} & (-1)^{n+1} \\ 3^n & (-1)^n \end{bmatrix}} \underbrace{\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{= \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 3^{n+1} - 2(-1)^{n+1} \\ 3^n - 2(-1)^n \end{bmatrix}$.

In particular, $a_n = 3^n - 2(-1)^n$.

(simplified solution) The eigenvalues of $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ are 3 and -1.

Looking back at our work above, we can see that a_n therefore must have a formula of the form $a_n = C_1 \cdot 3^n + C_2 \cdot (-1)^n$ for some unknown constants C_1, C_2 which we still need to figure out

Using the two initial conditions, we get two equations:

$(a_0=)\ C_1 + C_2 = -1$, $(a_1 =)\ 3C_1 - C_2 = 5$.

Solving, we find $C_1 = 1$ and $C_2 = -2$ so that, in conclusion, $a_n = 3^n - 2(-1)^n$.

   (d) It follows from the explicit formula that $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_1 = 0$.

To be very precise. To see that $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = 3$ we can observe that (by dividing each term by 3^n)

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1} - 2(-1)^{n+1}}{3^n - 2(-1)^n} = \frac{3 + 2\left(-\frac{1}{3}\right)^n}{1 - 2\left(-\frac{1}{3}\right)^n} \quad \text{as } n \to \infty \quad \frac{3+0}{1-0} = 3.$$

Definition 111. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = r_1 a_{n+d-1} + r_2 a_{n+d-2} + \ldots + r_d a_n$$

is called C-**finite** (or, **constant recursive**) of order d.

For instance. For the Fibonacci numbers, $d = 2$ and $r_1 = r_2 = 1$.

In matrix-vector form. $\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 & r_2 & \cdots & r_{d-1} & r_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$

By the same reasoning as in the previous example, C-finite sequences always have and explicit formula (called Binet-like because it is known as the Binet formula for the Fibonacci numbers):

Theorem 112. (generalized Binet formula) Suppose the recursion matrix M has distinct eigenvalues $\lambda_1, ..., \lambda_d$. Then

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + ... + C_d \lambda_d^n$$

for certain numbers $C_1, ..., C_d$. If, in addition, the eigenvalue λ_1 is larger in absolute value than the others (i.e. $|\lambda_1| > |\lambda_j|$ for $j = 2, 3, ..., d$) and $C_1 \neq 0$, then

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda_1.$$

How to see the limit. The limit is a consequence of $a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + ... + C_d \lambda_d^n$ because, for large n, the term $C_1 \lambda_1^n$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{C_1 \lambda_1^{n+1} + C_2 \lambda_2^{n+1} + ... + C_d \lambda_d^{n+1}}{C_1 \lambda_1^n + C_2 \lambda_2^n + ... + C_d \lambda_d^n} = \frac{C_1 \lambda_1 + C_2 \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + ... + C_d \lambda_d \left(\frac{\lambda_d}{\lambda_1}\right)^n}{C_1 + C_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + ... + C_d \left(\frac{\lambda_d}{\lambda_1}\right)^n} \stackrel{n \to \infty}{\longrightarrow} \frac{C_1 \lambda_1}{C_1} = \lambda_1.$$

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1 + \sqrt{5}}{2}$, $\lambda_2 = \frac{1 - \sqrt{5}}{2}$, and $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$. See Theorem 115.

Comment. More care is needed in the case that eigenvalues are repeated. Also, we need to be careful if there are several roots of the same absolute value. Consider, for instance, the case $a_n = 2^n + (-2)^n$. Can you see that, in this case, the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ doesn't even exist?

Example 113. Consider the sequence a_n defined by $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ and $a_0 = 0$, $a_1 = -2$, $a_2 = 2$.

  (a) Determine the first few terms of the sequence.

  (b) Find an explicit (Binet-like) formula for a_n.

  (c) Determine $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Solution.

  (a) $0, -2, 2, 10, 50, 178, 602, 1930, 6050, ...$

    Note that this sequence is C-finite of order 3.

  (b) The recursion can be translated to $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

    Expanding by the 2nd row: $\begin{vmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -6 \\ 1 & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 4-\lambda & -6 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6$

    The eigenvalues of the transition matrix are the roots of this polynomial: $\lambda = -1, 2, 3$

    [You will not be asked to find roots of cubic polynomials by hand.]

    Hence, $a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot 3^n$ and we only need to figure out the two unknowns C_1, C_2, C_3.

    Using the three initial conditions, we get three equations:

    $(a_0 =)$ $C_1 + C_2 + C_3 = 0$, $(a_1 =)$ $-C_1 + 2C_2 + 3C_3 = -2$, $(a_2 =)$ $C_1 + 4C_2 + 9C_3 = 2$.

    Solving, we find $C_1 = 1$, $C_2 = -2$ and $C_3 = 1$ so that, in conclusion, $a_n = (-1)^n - 2 \cdot 2^n + 3^n$.

    Comment. Do you see how we might have found the characteristic polynomial directly from the recursion?

  (c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

    Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_3 = 0$.

Example 114. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$.

Solution. The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{5}$. Hence, $a_n = C_1(1+\sqrt{5})^n + C_2(1-\sqrt{5})^n$ for certain numbers C_1, C_2.

[Note that we cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{5})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]
Therefore, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1}F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Let us work out an explicit formula for the Fibonacci numbers. This works exactly as in our previous examples, except that the eigenvalues involve square roots.

Theorem 115. (Binet's formula) $F_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2} \right)^n - \left(\dfrac{1-\sqrt{5}}{2} \right)^n \right]$

Proof.

- We already observed that thee recurrence $F_{n+2} = F_{n+1} + F_n$ translates into $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$
  and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $M = PDP^{-1}$ with

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

  Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = M^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix}$$

- Hence, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, which is the claimed formula. $\qquad\square$

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}}\lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$.

Back to the quotients of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Comment. It follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?