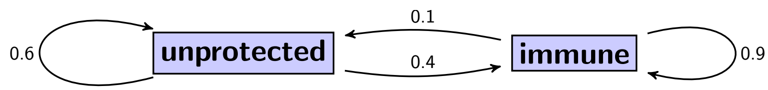
Application: Markov chains

Example 101. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.

What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)

y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

Powers of the transition matrix. Note that $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. In other words, M^n describes the transition over n years. In particular, the powers of M are the key to understanding what happens in this model over time.

Equilibrium. $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$. In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{1+4} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$.

Hence, the immunization rate in the long term equilibrium is $4/5 = 80\%$.

[Ponder about why this is a reasonable value!]

Comment. What's the other eigenvalue of the transition matrix? No need to compute the characteristic polynomial: we can easily see that it is $0.5 = 0.6 \cdot 0.9 - 0.1 \cdot 0.4$ because the product of the eigenvalues equals the determinant!

The 0.5-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Comment. Will the immunization rate always stabilize and approach the long term equilibrium? Yes! This is a consequence of the other eigenvalue of the transition matrix satisfying $|0.5| < 1$. If we start in state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 4 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1^n \cdot a\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (0.5)^n \cdot b\begin{bmatrix} -1 \\ 1 \end{bmatrix} \underset{\text{as } n\to\infty}{\longrightarrow} a\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Random comment. A rule of thumb is that a tetanus vaccination begins to wear off after about 10 years (somewhat in line with the 0.1 transition proportion in this example). However, the tetanus immunzation rate in the United States appears to be considerable less than the 80% we found in this (awfully simplistic) example.

https://www.cdc.gov/mmwr/preview/mmwrhtml/mm5940a3.htm

Example 102. Compute M^n for $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$.

Solution. In Example 99, we computed that $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$ had powers $A^n = \frac{1}{5}\begin{bmatrix} 10^n + 4 \cdot 5^n & 10^n - 5^n \\ 4 \cdot 10^n - 4 \cdot 5^n & 4 \cdot 10^n + 5^n \end{bmatrix}$.

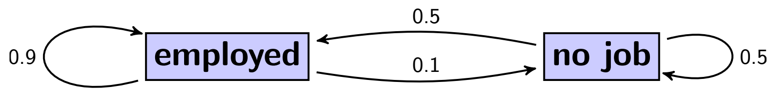
Since $M = \frac{1}{10}A$, this implies that $M^n = \frac{1}{10^n}A^n = \frac{1}{5}\begin{bmatrix} 1 + 4 \cdot 0.5^n & 1 - 0.5^n \\ 4 - 4 \cdot 0.5^n & 4 + 0.5^n \end{bmatrix}$.

Note that $M^n \to \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$ as $n \to \infty$. This reflects the fact that $\frac{1}{5}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the long term equilibrium.

Armin Straub
straub@southalabama.edu

Example 103. (extra) Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Solution. Let x_t and y_t be the proportions of those employed and unemployed. Proceeding, as in the previous example, the transition matrix is $M = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$.



The 1-eigenspace of M, that is $\operatorname{null}\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix}\right)$, has basis $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$. The corresponding equilibrium is $\frac{1}{5+1}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

In particular, the unemployment rate in the long term equilibrium is $1/6 \approx 16.7\%$.

Example 104. Which of the following are true for all square matrices A?

- Is it true that A^T has the same eigenvalues as A?
- Is it true that A^T has the same eigenspaces as A?
- Is it true that A^T has the same characteristic polynomial as A?

Solution. True. False. True.

First, note that the characteristic polynomial $\det(A - \lambda I)$ is the same as $\det(A^T - \lambda I)$. [Make sure you can fill in the details of why this is the case!] Hence, the eigenvalues (which are the roots of the characteristic polynomial) are also the same for A and A^T.

On the other hand, A^T and A in general have very different eigenspaces. Take, for instance, the matrix $A = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ from Example 101. Then both A and A^T have eigenvalues $\lambda = 0.5, 1$.

However, the 1-eigenspace of A is spanned by $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, while the 1-eigenspace of A^T is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 105. Show that a Markov matrix A (so that the columns of A sum to 1) always has eigenvalue 1.

Solution. This follows because the transpose A^T always has $\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$ as an 1-eigenvector (by virtue of the rows of A^T summing to 1). [Make sure that makes sense!]

By the previous example, A must also have eigenvalue 1 (but we have no idea what a 1-eigenvector is until we compute it).