

**Example 95. (review)** Let  $A$  be a symmetric  $2 \times 2$  matrix with 2-eigenvector  $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$  and  $\det(A) = 8$ . Determine the second eigenvalue and a corresponding eigenvector.

Further, diagonalize  $A$  as  $A = PDP^T$ .

**Solution.**  $A$  has  $8/2 = 4$ -eigenvector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  (chosen to be orthogonal to  $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$  since, by the spectral theorem, symmetric matrices have orthogonal eigenspaces).

Hence,  $A = PDP^T$  with  $D = \begin{bmatrix} 2 & \\ & 4 \end{bmatrix}$  and  $P = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ .

## Powers of matrices

**Example 96. (warmup)** Consider  $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ .

- What are the eigenspaces?
- What are  $A^{-1}$  and  $A^{100}$ ? What is  $A^n$ ?

**Solution.**

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is a  $-2$ -eigenvector, and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is a  $3$ -eigenvector. In other words, the  $-2$ -eigenspace is  $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$  and the  $3$ -eigenspace is  $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ .
- $A^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$  and  $A^{100} = \begin{bmatrix} (-2)^{100} & 0 \\ 0 & 3^{100} \end{bmatrix} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$ . In general,  $A^n = \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix}$ .

**Comment.** Algebraically, the map  $v \mapsto Av$  looks very simple. However, notice that it is not so easy to say what happens to, say,  $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  geometrically. That is because two things are happening: part of the vector  $v$  is scaled by  $-2$ , the other part is scaled by  $3$ .

**Example 97.** If  $A$  has  $\lambda$ -eigenvector  $v$ , then what can we say about  $A^2$ ?

**Solution.**  $A^2$  has  $\lambda^2$ -eigenvector  $v$ .

[Indeed,  $A^2v = A(Av) = A(\lambda v) = \lambda Av = \lambda^2v$ . This is even easier in words: multiplying  $v$  with  $A$  has the effect of scaling it by  $\lambda$ ; hence, multiplying it with  $A^2$  scales it by  $\lambda^2$ .]

**Important comment.** Similarly,  $A^{100}$  has  $\lambda^{100}$ -eigenvector  $v$ .

**Example 98.** If a matrix  $A$  can be diagonalized as  $A = PDP^{-1}$ , what can we say about  $A^n$ ?

**Solution.** First, note that  $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$ . Likewise,  $A^n = PD^nP^{-1}$ .

[The point being that  $D^n$  is trivial to compute because  $D$  is diagonal.]

**In particular.**  $A^{-1} = PD^{-1}P^{-1}$

**Important comment.** In the previous example, we observed that, if  $A$  has  $\lambda$ -eigenvector  $v$ , then  $A^n$  has  $\lambda^n$ -eigenvector  $v$ . Note that this is also expressed in  $A^n = PD^nP^{-1}$ , because the latter is a diagonalization of  $A^n$ . The diagonalization shows that  $A^n$  and  $A$  have the same eigenvectors (since we can use the same matrix  $P$ ) and that the eigenvalues of  $A^n$  are the  $n$ -th powers of the eigenvalues of  $A$  (which are the entries of the diagonal matrix  $D$ ).

**(computing matrix powers)** If  $A$  is a square matrix with diagonalization  $A = PDP^{-1}$ , then

$$A^n = PD^nP^{-1}.$$

**Example 99.** Let  $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$ . Compute  $A^n$ .

**Solution.** First, we diagonalize:  $A = PDP^{-1}$  with  $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$  and  $D = \begin{bmatrix} 10 & \\ & 5 \end{bmatrix}$ . (Fill in the details!)

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 10^n & \\ & 5^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 10^n & 10^n \\ -4 \cdot 5^n & 1 \cdot 5^n \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10^n + 4 \cdot 5^n & 10^n - 5^n \\ 4 \cdot 10^n - 4 \cdot 5^n & 4 \cdot 10^n + 5^n \end{bmatrix}$$

**Check.** Verify the cases  $n=0$  ( $A^0=I$ ) and  $n=1$ .

**Example 100. (extra)** Let  $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ . Determine  $A^n$ .

**Solution.** We first repeat our work from Example 17 to find a diagonalization of  $A$ :

By expanding by the second column, we find that the characteristic polynomial  $\det(A - \lambda I)$  is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda) - 2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are  $\lambda=2$  (with multiplicity 2) and  $\lambda=5$ .

- $\lambda=5$ :  $\text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \stackrel{\text{RREF}}{=} \text{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$
- $\lambda=2$ :  $\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \stackrel{\text{RREF}}{=} \text{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

We therefore have the diagonalization  $A = PDP^{-1}$  with  $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ ,  $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

[Keep in mind that other choices for  $P$  and  $D$  exist.]

With some labor (do it!), we find  $P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix}$ .

It follows that

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 \cdot 5^n & 0 & -2^n \\ 2 \cdot 5^n & 2^n & 0 \\ 5^n & 0 & 2^n \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 \cdot 5^n + 2^n & 0 & 2 \cdot 5^n - 2 \cdot 2^n \\ 2 \cdot 5^n - 2 \cdot 2^n & 3 \cdot 2^n & 2 \cdot 5^n - 2 \cdot 2^n \\ 5^n - 2^n & 0 & 5^n + 2 \cdot 2^n \end{bmatrix}. \end{aligned}$$

**Check.** Notice that it is particularly easy to verify the cases  $n=0$  ( $A^0=I$ ) and  $n=1$ .