

## The fundamental theorem

**Review.** The four **fundamental subspaces** associated with a matrix  $A$  are

$$\text{col}(A), \quad \text{row}(A), \quad \text{null}(A), \quad \text{null}(A^T).$$

Note that  $\text{row}(A) = \text{col}(A^T)$ . (In particular, we usually write vectors in  $\text{row}(A)$  as column vectors.)

**Comment.**  $\text{null}(A^T)$  is called the **left null space** of  $A$ .

Why that name? Recall that, by definition  $\mathbf{x}$  is in  $\text{null}(A) \iff A\mathbf{x} = \mathbf{0}$ .

Likewise,  $\mathbf{x}$  is in  $\text{null}(A^T) \iff A^T\mathbf{x} = \mathbf{0} \iff \mathbf{x}^T A = \mathbf{0}$ .

[Recall that  $(AB)^T = B^T A^T$ . In particular,  $(A^T \mathbf{x})^T = \mathbf{x}^T A$ , which is what we used in the last equivalence.]

**Review.** The **rank** of a matrix is the number of pivots in its RREF.

Equivalently, as showcased in the next result, the rank is the dimension of either the column or the row space.

### Theorem 35. (Fundamental Theorem of Linear Algebra, Part I)

Let  $A$  be an  $m \times n$  matrix of **rank**  $r$ .

- $\dim \text{col}(A) = r$  (subspace of  $\mathbb{R}^m$ )
- $\dim \text{row}(A) = r$  (subspace of  $\mathbb{R}^n$ )  $\text{row}(A) = \text{col}(A^T)$
- $\dim \text{null}(A) = n - r$  (subspace of  $\mathbb{R}^n$ )
- $\dim \text{null}(A^T) = m - r$  (subspace of  $\mathbb{R}^m$ )

**Example 36.** Let  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ . Determine bases for all four fundamental subspaces.

**Solution.** Make sure that, for such a simple matrix, you can see all of these that at a glance!

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}, \quad \text{row}(A) = \text{span} \left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}, \quad \text{null}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \quad \text{null}(A^T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**Important observation.** The basis vectors for  $\text{row}(A)$  and  $\text{null}(A)$  are orthogonal!  $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for  $\text{col}(A)$  and  $\text{null}(A^T)$ :  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$  and  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

**Always.** Vectors in  $\text{null}(A)$  are orthogonal to vectors in  $\text{row}(A)$ . In short,  $\text{null}(A)$  is orthogonal to  $\text{row}(A)$ .

**Why?** Suppose that  $\mathbf{x}$  is in  $\text{null}(A)$ . That is,  $A\mathbf{x} = \mathbf{0}$ . But think about what  $A\mathbf{x} = \mathbf{0}$  means (row-product rule). It means that the inner product of every row with  $\mathbf{x}$  is zero. Which implies that  $\mathbf{x}$  is orthogonal to the row space.

### Theorem 37. (Fundamental Theorem of Linear Algebra, Part II)

- $\text{null}(A)$  is orthogonal to  $\text{row}(A)$ . (both subspaces of  $\mathbb{R}^n$ )

Note that  $\dim \text{null}(A) + \dim \text{row}(A) = n$ . Hence, the two spaces are orthogonal complements.

- $\text{null}(A^T)$  is orthogonal to  $\text{col}(A)$ .

Again, the two spaces are orthogonal complements. (This is just the first part with  $A$  replaced by  $A^T$ .)

**Example 38.** Let  $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$ . Check that  $\text{null}(A)$  and  $\text{row}(A)$  are orthogonal complements.

**Solution.**

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix} \xrightarrow[R_3 - 3R_1 \Rightarrow R_3]{R_2 - 2R_1 \Rightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & -3 & -9 \end{bmatrix} \xrightarrow[R_3 - \frac{3}{2}R_2 \Rightarrow R_3]{R_3 - \frac{3}{2}R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{2}R_2 \Rightarrow R_2]{-\frac{1}{2}R_2 \Rightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 - R_2 \Rightarrow R_1]{R_1 - R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence,  $\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ ,  $\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ .

$\text{null}(A)$  and  $\text{row}(A)$  are indeed orthogonal, as certified by:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$$

In fact,  $\text{null}(A)$  and  $\text{row}(A)$  are orthogonal complements because the dimensions add up to  $2 + 2 = 4$ .

In particular,  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$  form a basis of all of  $\mathbb{R}^4$ .

**Example 39. (extra)** Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that  $\text{null}(A)$  and  $\text{row}(A)$  (as well as  $\text{null}(A^T)$  and  $\text{col}(A)$ ) are orthogonal complements.

**Partial solution.** One can almost see that  $\text{rank}(A) = 3$ . Hence, the dimensions of the fundamental subspaces are ...