Example 23. (review) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Recall that $(AB)^T = B^T A^T$. This reflects the fact that, in the column-centric versus the row-centric interpretation of matrix multiplication, the order of the matrices is reversed.

Comment. When working with complex numbers, the fundamental role is not played by the transpose but by the **conjugate transpose** instead (we'll see that in our discussion of orthogonality): $A^* = \overline{A^T}$.

For instance, if $A = \begin{bmatrix} 1-3i & 5i \\ 2+i & 3 \end{bmatrix}$, then $A^* = \begin{bmatrix} 1+3i & 2-i \\ -5i & 3 \end{bmatrix}$.

Orthogonality

The inner product and distances

Definition 24. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 25. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

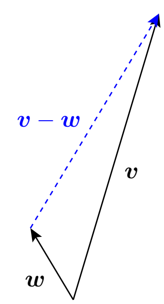
Definition 26.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 27. For instance, in $\mathbb{R}^2$, $\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\|\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}\right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 28. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

Comment. This is a vector version of $(x - y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

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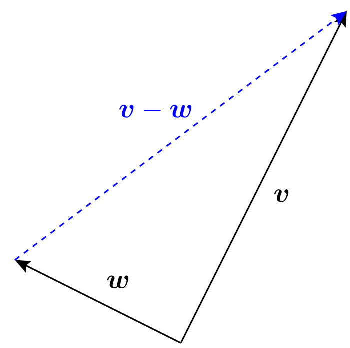
Definition 29. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v}\cdot\boldsymbol{w}=0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$$\iff \|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2=\underbrace{\|\boldsymbol{v}-\boldsymbol{w}\|^2}_{\substack{=\|\boldsymbol{v}\|^2-2\boldsymbol{v}\cdot\boldsymbol{w}+\|\boldsymbol{w}\|^2\\ \text{(by previous example)}}}$$

$$\iff -2\boldsymbol{v}\cdot\boldsymbol{w}=0$$

$$\iff \boldsymbol{v}\cdot\boldsymbol{w}=0$$

Definition 30. We say that two subspaces V and W of $\mathbb{R}^n$ are **orthogonal** if and only if every vector in V is orthogonal to every vector in W.

The **orthogonal complement** of V is the space $V^\perp$ of all vectors that are orthogonal to V.

Exercise. Show that the orthogonal complement is indeed a vector space. Alternatively, this follows from our discussion in the next example which leads to Theorem 32. Namely, every space V can be written as $V=\operatorname{col}(A)$ for a suitable matrix A (for instance, we can choose the columns of A to be basis vectors of V). It then follows that $V^\perp=\operatorname{null}(A^T)$ (which is clearly a space).

Example 31. Determine a basis for the orthogonal complement of $V=\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$.

Solution. The orthogonal complement $V^\perp$ consists of all vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ that are orthogonal to $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\\2\end{bmatrix}$.

Using the dot product, this means we must have $\begin{bmatrix}1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ as well as $\begin{bmatrix}3\\1\\2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$.

Note that this is equivalent to the equations $1x_1+2x_2+1x_3=0$ and $3x_1+1x_2+2x_3=0$.

In matrix-vector form, these two equations combine to $\begin{bmatrix}1&2&1\\3&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

This is the same as saying that $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ has to be in $\operatorname{null}\left(\begin{bmatrix}1&2&1\\3&1&2\end{bmatrix}\right)$. This means that $V^\perp=\operatorname{null}\left(\begin{bmatrix}1&2&1\\3&1&2\end{bmatrix}\right)$.

[Note that we have done no computations up to this point! Instead, we have derived Theorem 32 below.]

We compute (fill in the work!) that $V^\perp=\operatorname{null}\left(\begin{bmatrix}1&2&1\\3&1&2\end{bmatrix}\right)\overset{\text{RREF}}{=}\operatorname{null}\left(\begin{bmatrix}1&0&3/5\\0&1&1/5\end{bmatrix}\right)=\operatorname{span}\left\{\begin{bmatrix}-3/5\\-1/5\\1\end{bmatrix}\right\}$.

Check. $\begin{bmatrix}-3/5\\-1/5\\1\end{bmatrix}$ is indeed orthogonal to both $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\\2\end{bmatrix}$.

Note. If $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ is orthogonal to both basis vectors $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\\2\end{bmatrix}$, then it is orthogonal to every vector in V.

Indeed, vectors in V are of the form $\boldsymbol{v}=a\begin{bmatrix}1\\2\\1\end{bmatrix}+b\begin{bmatrix}3\\1\\2\end{bmatrix}$ and we have $\boldsymbol{v}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=a\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{=0}+b\underbrace{\begin{bmatrix}3\\1\\2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{=0}=0$.

Just to make sure. Why is it geometrically clear that the orthogonal complement of V is 1-dimensional?

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The following theorem follows by the same reasoning that we used in the previous example.

In that example, we started with $V = \operatorname{col}\left(\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ and found that $V^{\perp} = \operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right)$.

Theorem 32. If $V = \operatorname{col}(A)$, then $V^{\perp} = \operatorname{null}(A^T)$.
In particular, if V is a subspace of $\mathbb{R}^n$ with $\dim(V) = r$, then $\dim(V^{\perp}) = n - r$.

For short. $\operatorname{col}(A)^{\perp} = \operatorname{null}(A^T)$

Note that the second part can be written as $\dim(V) + \dim(V^{\perp}) = n$.

To see that this is true, suppose we choose the columns of A to be a basis of V. If V is a subspace of $\mathbb{R}^n$ with $\dim(V) = r$, then A is a $r \times n$ matrix with r pivot columns. Correspondingly, A^T is a $n \times r$ matrix with r pivot rows. Since $n \geqslant r$ there are $n - r$ free variables when computing a basis for $\operatorname{null}(A^T)$. Hence, $\dim(V^{\perp}) = n - r$.

Example 33. Suppose that V is spanned by 3 linearly independent vectors in $\mathbb{R}^5$. Determine the dimension of V and its orthogonal complement $V^{\perp}$.

Solution. This means that $\dim V = 3$. By Theorem 32, we have $\dim V^{\perp} = 5 - 3 = 2$.

Example 34. Determine a basis for the orthogonal complement of (the span of) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Solution. Here, $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ and we are looking for the orthogonal complement $V^{\perp}$.

Since $V = \operatorname{col}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$, it follows from Theorem 32 that $V^{\perp} = \operatorname{null}([\,1\ \ 2\ \ 1\,])$.

Computing a basis for $\operatorname{null}([\,1\ \ 2\ \ 1\,])$ is easy since $[\,1\ \ 2\ \ 1\,]$ is already in RREF.

Note that the general solution to $[\,1\ \ 2\ \ 1\,]\boldsymbol{x} = 0$ is $\begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $V^{\perp} = \operatorname{null}([\,1\ \ 2\ \ 1\,])$ therefore is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Check. We easily check (do it!) that both of these are indeed orthogonal to the original vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

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