Example 18. (extra practice) Diagonalize, if possible, the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution. For instance, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & & \\ & 2 & \\ & & 2 \end{bmatrix}$. B is not diagonalizable.

For instance, $C = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$.

Review: Computing determinants using cofactor expansion

Review. Let A be an $n \times n$ matrix. The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\det(A) \neq 0 \iff A \text{ is invertible}$$
$$\iff Ax = b \text{ has a (unique) solution } x \text{ (for all } b)$$
$$\iff Ax = 0 \text{ is only solved by } x = 0$$

Example 19. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example 20. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & \\ -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} - \\ 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} + \\ 3 & -1 \\ 2 & 0 \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1 \cdot (-1) - 2 \cdot (-1) + 0 = 1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

The ± 1 is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$.

Solution. We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} - \\ 3 & 2 \\ 2 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 0 \\ + \\ 2 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ 3 & 2 \\ - \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

Example 21. Compute $\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix}$.

Solution. We can expand by the second column:

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix} = -0\begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 2 & 8 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

[Of course, you don't have to spell out the 3×3 matrices that get multiplied with 0.]

We can compute the remaining 3×3 matrix in any way we prefer. One option is to expand by the first column:

$$2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} = 2\left(+1\begin{vmatrix} 2 & 1 \\ 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}\right) = 2(1 \cdot 2 + 2 \cdot (-5)) = -16$$

Comment. For cofactor expansion, choosing to expand by the second column is the best choice because this column has more zeros than any other column or row.

The determinant of a triangular matrix is the product of the diagonal entries.

Why? Can you explain this (you can use the next example) using cofactor expansion?

Example 22. Compute $\begin{vmatrix} 1 & 0 & 3 & -1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$.

Solution. Since the matrix is (upper) triangular, $\begin{vmatrix} 1 & 0 & 3 & -1 \\ & 3 & 1 & 5 \\ & & -2 & 1 \\ & & & 5 \end{vmatrix} = 1 \cdot 3 \cdot (-2) \cdot 5 = -30$.

Review.

- Effect of row (or column) operations on determinant.

- $\det(AB) = \det(A)\det(B)$

- In particular, the LU decomposition provides us with a way to compute determinants:
  If $A = LU$, then $\det(A) = \det(L)\det(U)$ and the latter determinants are just products of diagonal entries (because both L and U are triangular).

  Comment. Unless a row swap is required, we can compute the LU decomposition of $A = LU$ using only row operations of the form $R_i + cR_j \Rightarrow R_i$ (those don't change the determinant!).
  In that case, the matrix L will have 1's on the diagonal. In particular, $\det(L) = 1$.
  Consequently, in that case, $\det(A) = \det(U)$.

  Practical comment. For larger matrices, cofactor expansion is a terribly inefficient way of computing determinants. Instead, Gaussian elimination (i.e. LU decomposition) is much more efficient.
  On the other hand, cofactor expansion is a good choice when working by hand with small matrices.