Midterm #1

MATH 316 — Linear Algebra II
Friday, Feb 16, 2024

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 31 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points)

(a) Find the least squares solution to $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 2 \end{bmatrix}$.

(b) Determine the least squares line for the data points $(2, -2), (1, 0), (1, 5), (-1, 2)$.

Solution. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 2 \end{bmatrix}$. Clearly, $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{bmatrix}$.

(a) Since $A^T A = \begin{bmatrix} 4 & 3 \\ 3 & 7 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, so the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ are

$$\begin{bmatrix} 4 & 3 \\ 3 & 7 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

Solving, we find that the least squares solution is $\hat{\boldsymbol{x}} = \frac{1}{19} \begin{bmatrix} 7 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(b) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 2 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Hence, the least squares line is $y = 2 - x$.

Problem 2. (2 points) Suppose A is a symmetric 2×2 matrix with 3-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\det(A) = 6$.

Then A has $\boxed{}$-eigenvector $\boxed{}$. Further, $A = PDP^T$ with $D = \boxed{}$ and $P = \boxed{}$.

Solution. Then A has 2-eigenvector $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$. Further, $A = PDP^T$ with $D = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$ and $P = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$.

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Problem 3. (9 points)

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ onto W.

(c) Determine the orthogonal projection of that same vector onto $W^\perp$.

(d) Determine the QR decomposition of the matrix $\begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$.

Solution.

(a) Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be the vectors spanning W. We first construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ using Gram–Schmidt (and then normalize afterwards):

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Normalizing, we obtain the orthonormal basis $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ for W.

(b) The orthogonal projection of $\boldsymbol{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ onto W is

$$\frac{\boldsymbol{v} \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 + \frac{\boldsymbol{v} \cdot \boldsymbol{q}_2}{\boldsymbol{q}_2 \cdot \boldsymbol{q}_2} \boldsymbol{q}_2 = \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}.$$

(Check: the error $\frac{2}{3}(1, -1, 2)^T$ is indeed orthogonal to W.)

(c) We can compute this as the error of the projection in the previous part: $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

(d) From the first part, we know that $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & -1/\sqrt{3} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$.

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Problem 4. (3 points) We want to find values for the parameters a, b, c such that $z = ax + bx^2 + c\ln(y)$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution. The equations $ax_i + bx_i^2 + c\ln(y_i) = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} x_1 & x_1^2 & \ln(y_1) \\ x_2 & x_2^2 & \ln(y_2) \\ x_3 & x_3^2 & \ln(y_3) \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{z}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution.

Problem 5. (3 points) Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

(a) A basis for $\operatorname{null}(A)$ is ______. A basis for $\operatorname{col}(A)$ is ______.

(b) $\dim\operatorname{col}(A) =$ ______, $\dim\operatorname{row}(A) =$ ______, $\dim\operatorname{null}(A) =$ ______, $\dim\operatorname{null}(A^T) =$ ______

Solution.

(a) A basis for $\operatorname{null}(A)$ is $\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$. A basis for $\operatorname{col}(A)$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) $\dim\operatorname{col}(A) = 2$, $\dim\operatorname{row}(A) = 2$, $\dim\operatorname{null}(A) = 3$, $\dim\operatorname{null}(A^T) = 0$

Problem 6. (8 points) Fill in the blanks.

(a) $\operatorname{null}(A)$ is the orthogonal complement of ______. $\operatorname{col}(A)$ is the orthogonal complement of ______.

(b) If A is a 5×7 matrix with rank 4, then $\dim\operatorname{col}(A) =$ ______ and $\dim\operatorname{null}(A) =$ ______.

(c) By definition, a matrix Q is orthogonal if and only if ______.

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(d) If Q is orthogonal, then $\det(Q)$ is ______.

(e) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to ______.

(f) For which matrices A is it true that $A^{-1} = A^T$? ______

(g) The projection matrix for orthogonally projecting onto $\mathrm{col}(A)$ is $P =$ ______.

If A has orthonormal columns, this simplifies to ______.

(h) Let W be the subspace of $\mathbb{R}^5$ of all solutions to $x_1 - x_3 + 2x_5 = 0$. $\dim W =$ ______ and $\dim W^{\perp} =$ ______.

Solution.

(a) $\mathrm{null}(A)$ is the orthogonal complement of $\mathrm{col}(A^T)$. $\mathrm{col}(A)$ is the orthogonal complement of $\mathrm{null}(A^T)$.

(b) If A is a 5×7 matrix with rank 4, then $\dim \mathrm{col}(A) = 4$ and $\dim \mathrm{null}(A) = 7 - 4 = 3$.

(c) By definition, a matrix Q is orthogonal if and only if Q is $n \times n$ (square) and Q has orthonormal columns.

(d) If Q is orthogonal, then $\det(Q)$ is ± 1.

(e) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to $\mathrm{null}(A^T)$.

(f) Orthogonal matrices.

[For a square matrix, $A^{-1} = A^T$ if and only if $A^T A = I$. Hence, $A^{-1} = A^T$ if and only if A is a square matrix with orthonormal columns. Such matrices are called orthogonal (a somewhat unfortunate terminology).]

(g) The projection matrix for orthogonally projecting onto $\mathrm{col}(A)$ is $P = A\,(A^T A)^{-1} A^T$.

If A has orthonormal columns (so that $A^T A = I$), this simplifies to $A A^T$.

(h) If W is the space of all solutions to $x_1 - x_3 + 2x_5 = 0$, then $\dim W = 4$ and $\dim W^{\perp} = 1$.

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(extra scratch paper)