Preparing for Midterm #1

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1.

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W.

(c) Determine the orthogonal projection of that same vector onto $W^{\perp}$.

(d) Determine the QR decomposition of the matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(e) Determine a basis for the orthogonal complement $W^{\perp}$.

Problem 2.

(a) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$ as $A = PDP^T$. (That is, find the matrices P and D.)

(b) Let A be a symmetric 2×2 matrix with 2-eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\det(A) = -6$. Diagonalize A as $A = PDP^T$.

Problem 3.

(a) Find the least squares solution to the system $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto the space $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$?

(c) Determine the least squares line for the data points $(-2, 1), (-1, 0), (0, 3), (2, 1)$.

(d) Determine the projection matrix P for orthogonally projecting onto W.

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Problem 4. Trying to find a relation between the quantities x and y, we measure the values

x	1	2	3	4
y	2	5	9	17

(a) We expect that y can be predicted as a linear function of the form $a+bx$. Find the best estimate for the coefficients. ["best" in the least squares sense]

(b) What changes if we suppose that y can be predicted as a quadratic function of the form $a+bx+cx^2$? Set up a linear system such that $[a,b,c]^T$ is a least squares solution.

Problem 5.

(a) Is it true that A^TA is always symmetric?

(b) If the columns of A are orthogonal, what can you say about A^TA?

(c) Note that $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Why is it incorrect that the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto span$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ is $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$? Explain!

(d) For which matrices A is it true that $A^{-1} = A^T$?

Problem 6.

(a) We want to find values for the parameters a,b,c such that $y = a + bx + \frac{c}{x}$ best fits some given points (x_1, y_1), $(x_2, y_2), \ldots$ Set up a linear system such that $[a,b,c]^T$ is a least squares solution.

(b) We want to find values for the parameters a,b such that $y = (a+bx)e^x$ best fits some given points (x_1, y_1), $(x_2, y_2), \ldots$ Set up a linear system such that $[a,b]^T$ is a least squares solution.

(c) We want to find values for the parameters a,b,c such that $z = a + bx - c\sqrt{y}$ best fits some given points (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a,b,c]^T$ is a least squares solution.

Problem 7. Let W be the subspace of $\mathbb{R}^4$ of all solutions to $x_1 + x_2 + x_3 - x_4 = 0$.

(a) Find a basis for W.

(b) Find a basis for the orthogonal complement $W^\perp$.

(c) Determine the orthogonal projection of $\boldsymbol{b} = (1,1,1,1)^T$ onto $W^\perp$.

(d) Determine the orthogonal projection of $\boldsymbol{b} = (1,1,1,1)^T$ onto W.

Problem 8. Suppose that A is a 3×5 matrix of rank 3.

(a) For each of the four fundamental subspaces of A, state which space it is a subspace of.

(b) What are the dimensions of all four fundamental subspaces?

(c) Which fundamental subspaces are orthogonal complements of each other?

(d) For the specific matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$, compute a basis for each fundamental subspace.

(e) Observe that $\operatorname{rank}(A) = 3$. Then, verify that all your predictions made in the first three parts do in fact hold.